# Cost of fault-tolerance in quantum computation

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INVENTEURS DU MONDE NUMÉRIQUE

# Quantum vs classical

**Feature 1:** Schrödinger equation replaces Newton's laws

State of a quantum system: Wave-function $|\psi\rangle \in \mathcal{H}$ with $\mathcal{H}$ a complex Hilbert space.

Dynamics: $$i\frac{d}{dt}|\psi\rangle = \mathbf{H}|\psi\rangle$$

with $\mathbf{H}$ a Hermitian operator defined on the Hilbert space $\mathcal{H}$.

Solution: $$|\psi(t)\rangle = \mathbf{U}_t|\psi_0\rangle$$

with $\mathbf{U}_t$ a unitary operator defined on the Hilbert space $\mathcal{H}$ and solution of

$$i\frac{d}{dt}\mathbf{U}_t = \mathbf{H}\mathbf{U}_t, \qquad \mathbf{U}_0 = \mathrm{Id}$$

In case of quantum bits (qubits) these unitary operations are called **logical gates**.

# Quantum vs classical

**Feature 2:** Entanglement and tensor product for composite systems $S_1, S_2, \cdots, S_n$

Hilbert space: $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \mathcal{H}_n$ instead of $\mathcal{H}_1 \times \mathcal{H}_2 \cdots \times \mathcal{H}_n$

Dimension: $D = d_1 \times d_2 \cdots \times d_n$ instead of $d_1 + d_2 \cdots + d_n$

Entanglement and its surprising properties: $|\psi_1\rangle \otimes |\psi_2\rangle + |\widetilde{\psi}_1\rangle \otimes |\widetilde{\psi}_2\rangle \neq |\Psi\rangle \otimes |\widetilde{\Psi}\rangle$

Main source of trouble for classical simulations: huge dimensional Hilbert space

Main resource for quantum computation: huge dimensional Hilbert space

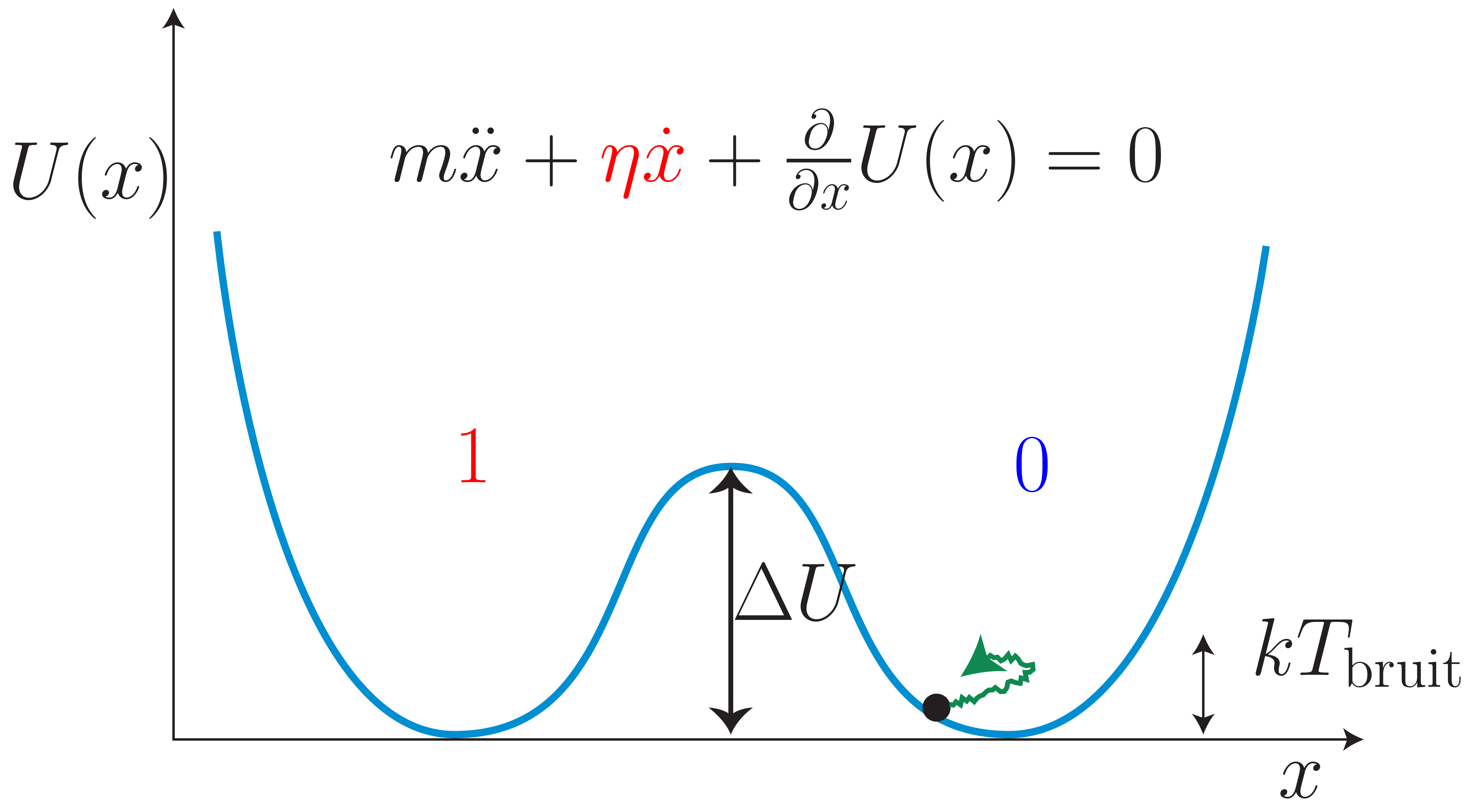
# Quantum vs classical

**Feature 3:** Randomness and irreversibility induced by measurement: any physical observable is represented by a Hermitian operator **O** with spectral decompisition $\mathbf{O} = \sum_\mu \lambda_\mu \mathbf{P}_\mu$ .

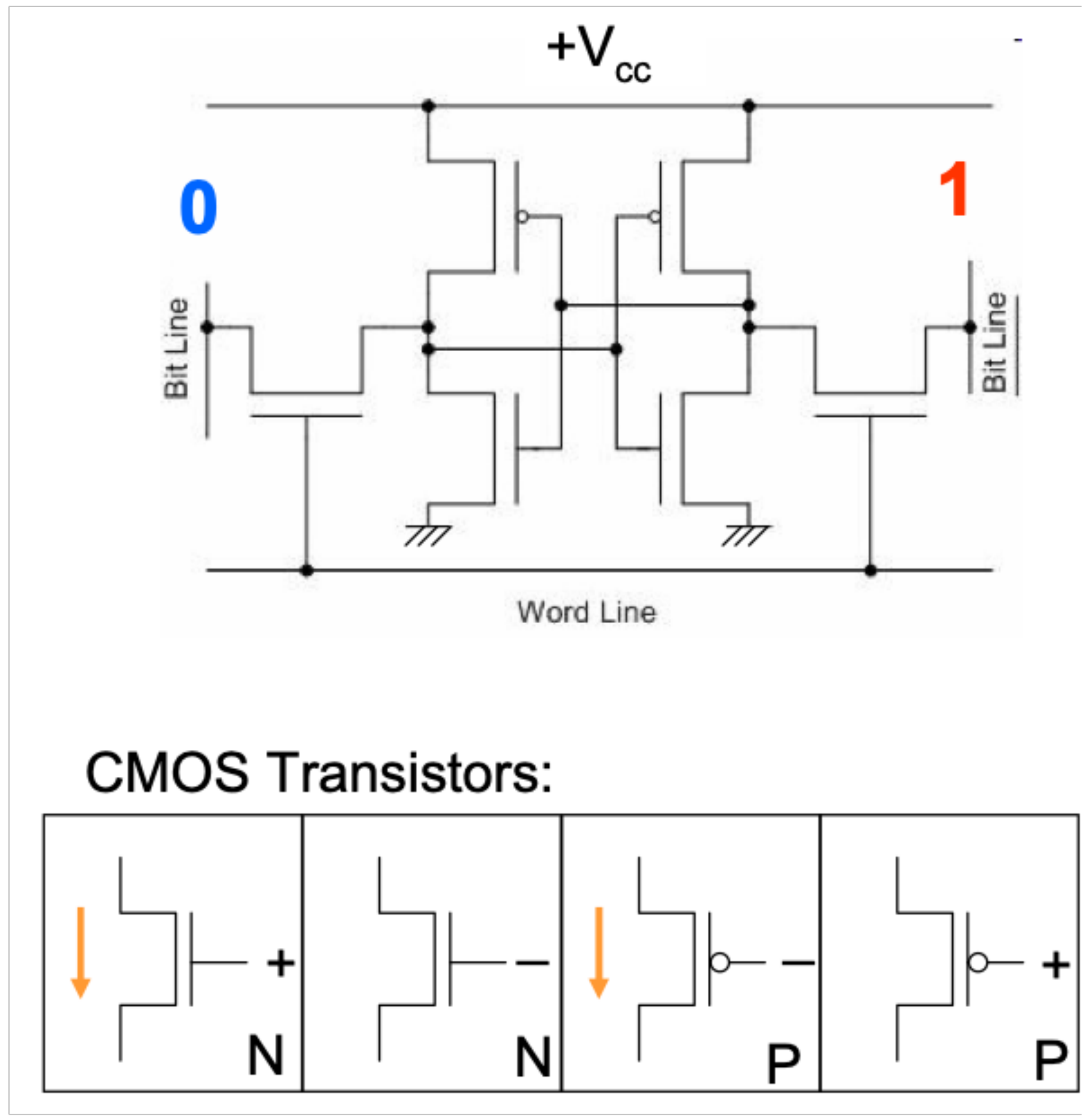
- Measurement outcome $\lambda_\mu$ with probability $p_\mu = \langle\psi|\mathbf{P}_\mu|\psi\rangle$ .
- Measurement backaction if outcome $\lambda_\mu$ : $|\psi_+\rangle = \dfrac{\mathbf{P}_\mu|\psi\rangle}{\sqrt{\langle\psi|\mathbf{P}_\mu|\psi\rangle}}$
- Puzzling for measurement and feedback control of quantum systems.
- Main resource for quantum communication and cryptography.

# Physics of information, classical bit

Classical bit: strongly dissipative bistable system




Typical SRAM cell




## Classical bit in state 0 or 1

- Strong dissipation (friction);
- $k_B T_{noise} \ll \Delta U$;

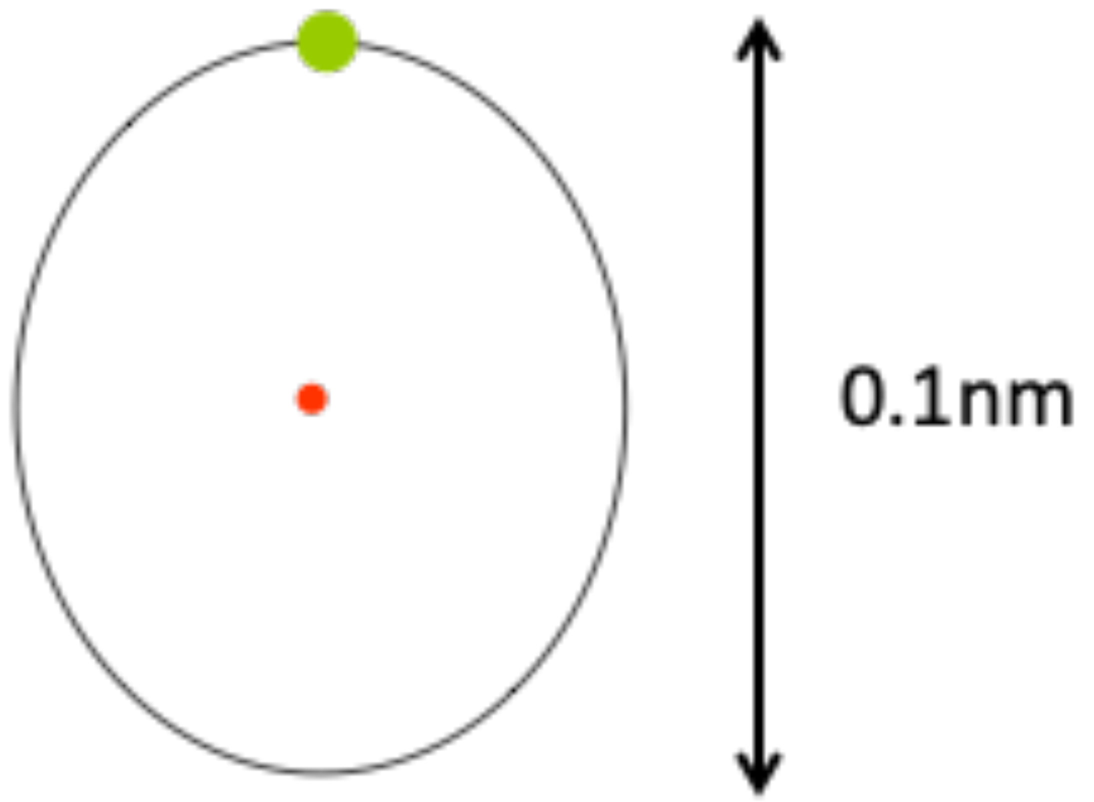
# Physics of information, quantum bit

$$\left(-\frac{\hbar^2}{2m}\Delta + U(x)\right)\psi_k(x) = E_k\psi_k(x), \qquad |0\rangle = \psi_0(x),\ |1\rangle = \psi_1(x)$$

Example:

$$\left(-\frac{\hbar^2}{2\mu}\triangle - \frac{e^2}{4\pi\varepsilon_0 r}\right)\psi_k(r,\theta,\varphi) = E_k\psi_k(r,\theta,\varphi)$$

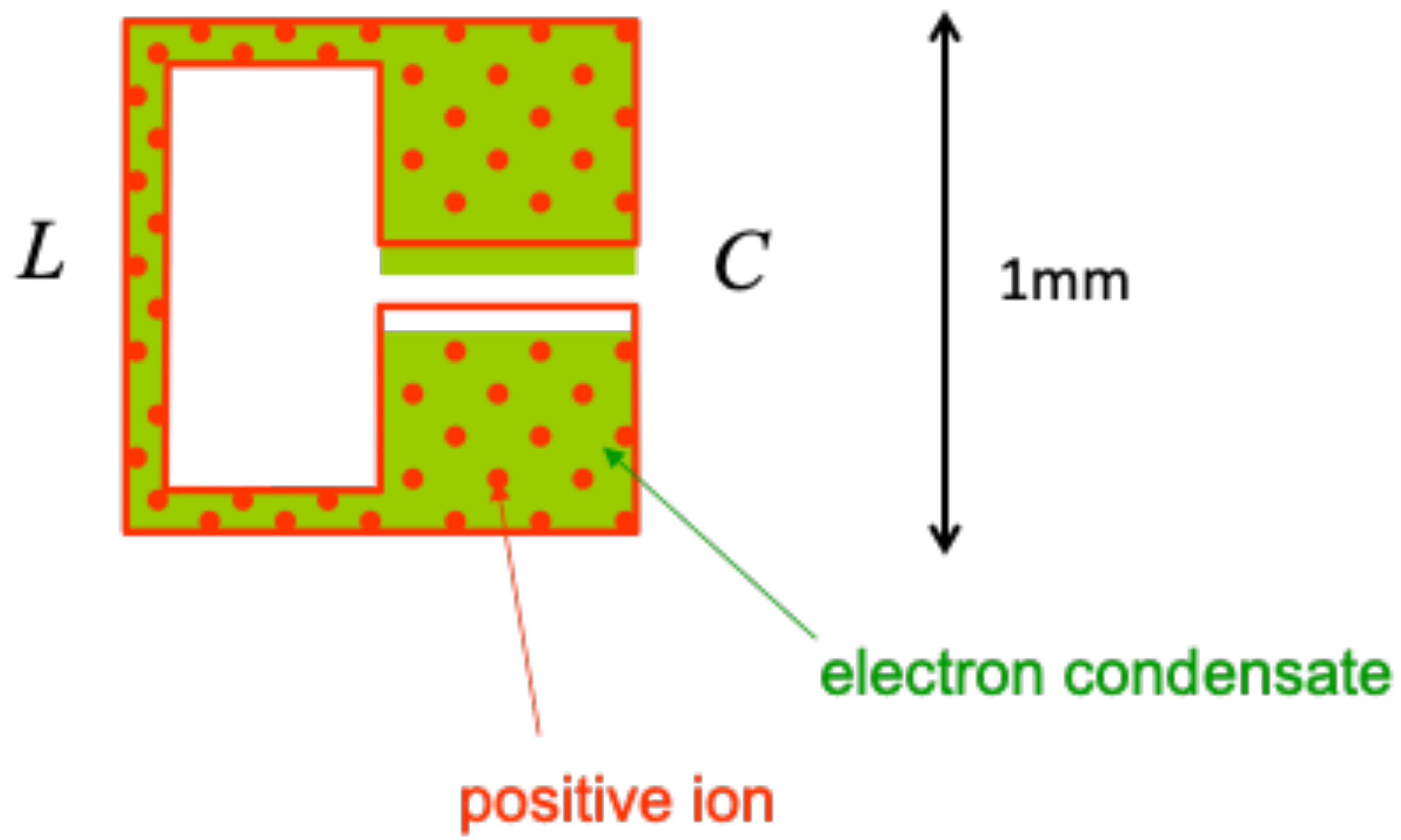
Hydrogen atom



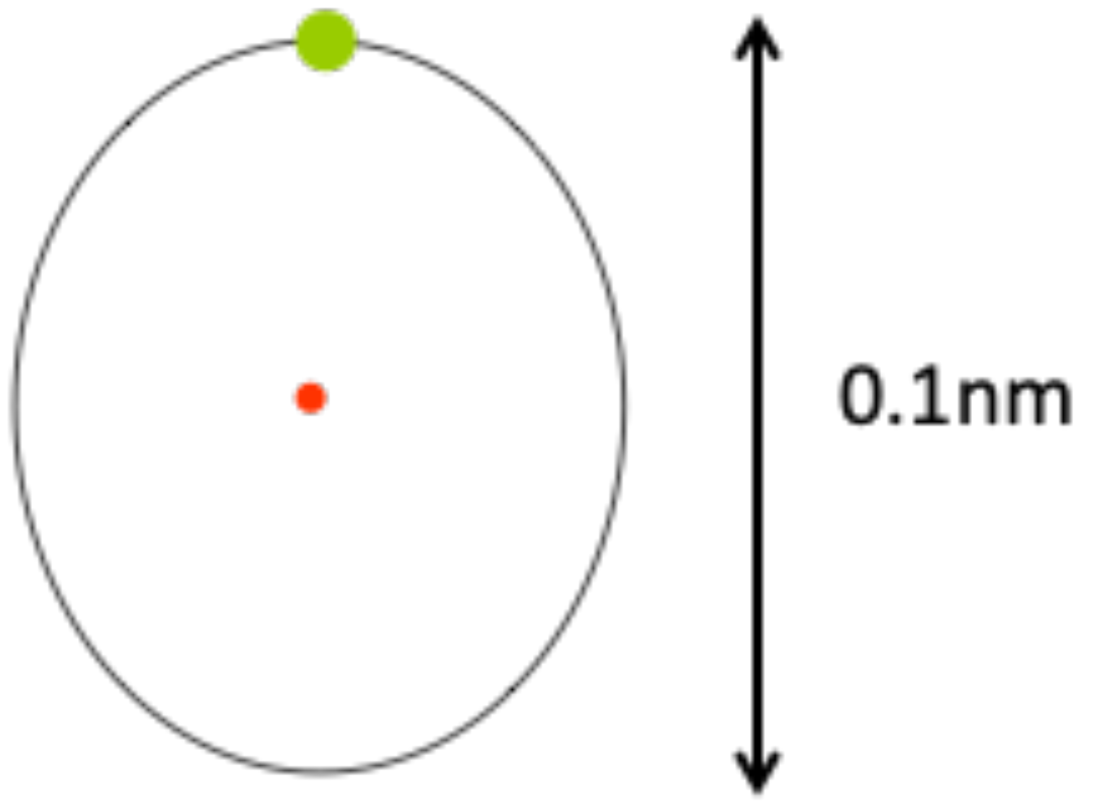

Qubit state: $c_0|0\rangle + c_1|1\rangle \in \mathrm{span}\{\psi_0, \psi_1\}$

# Superconducting circuits as quantum systems

Superconducting LC oscillator



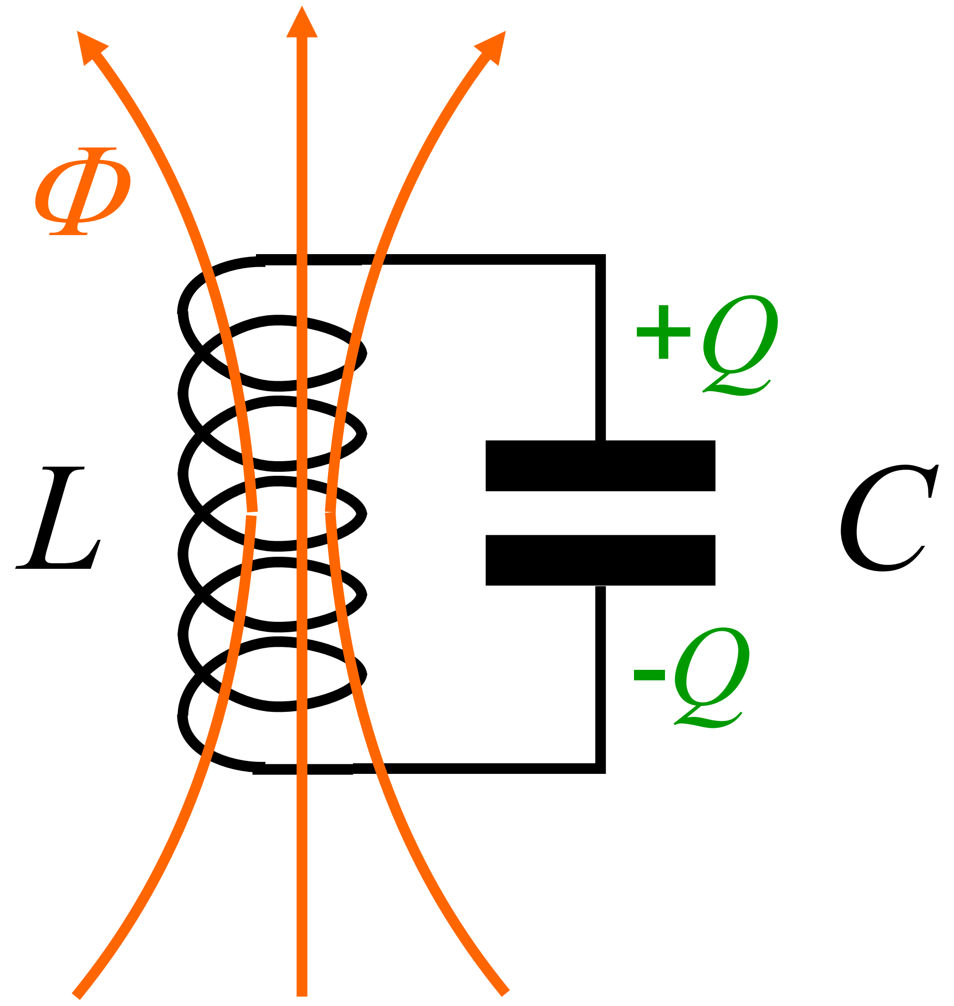

Hydrogen atom

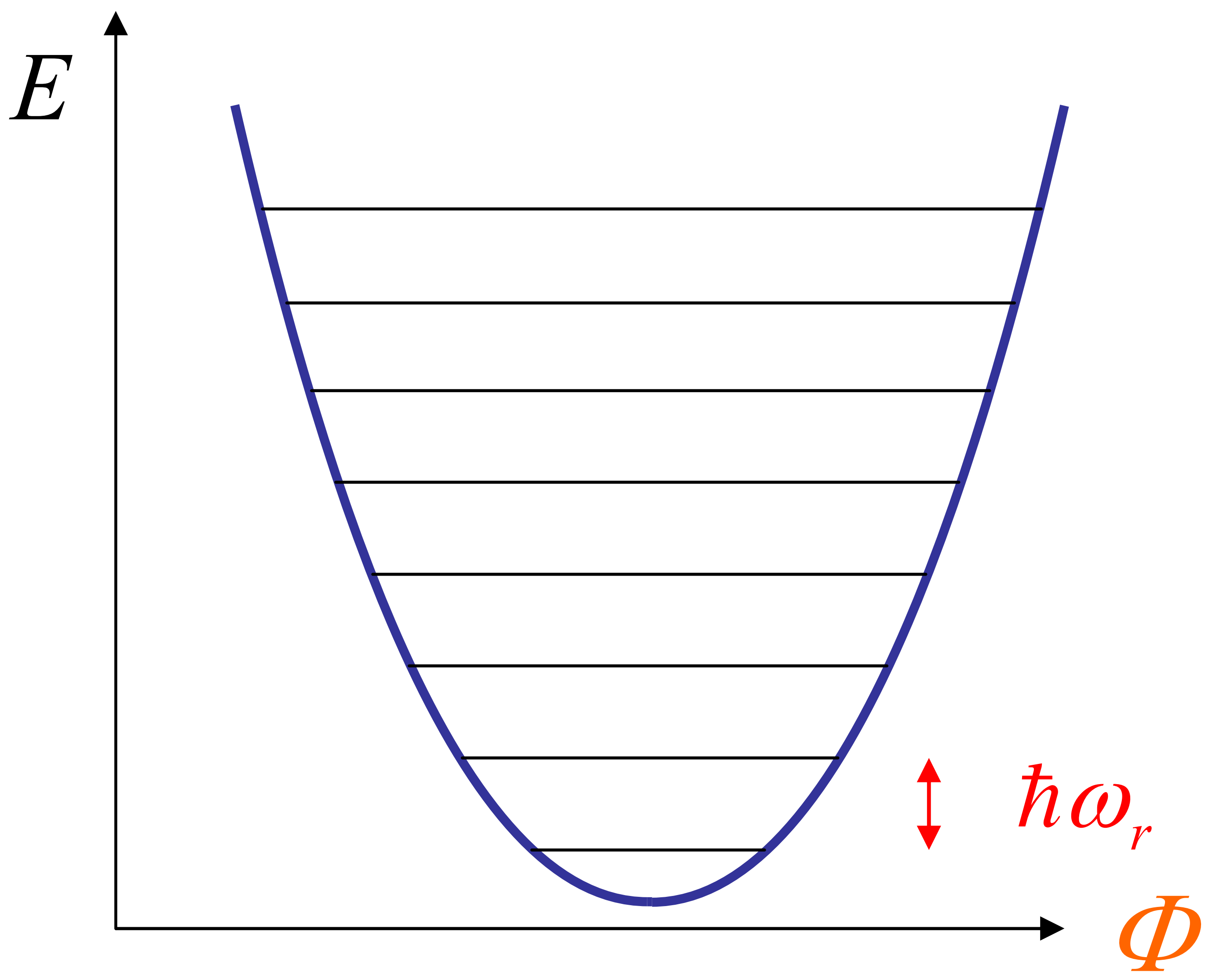



**Quantum regime:** low temperature of about 20mK for oscillation frequency 5-10 GHz (microwave regime).

Courtesy of Michel Devoret

# Quantum LC oscillator



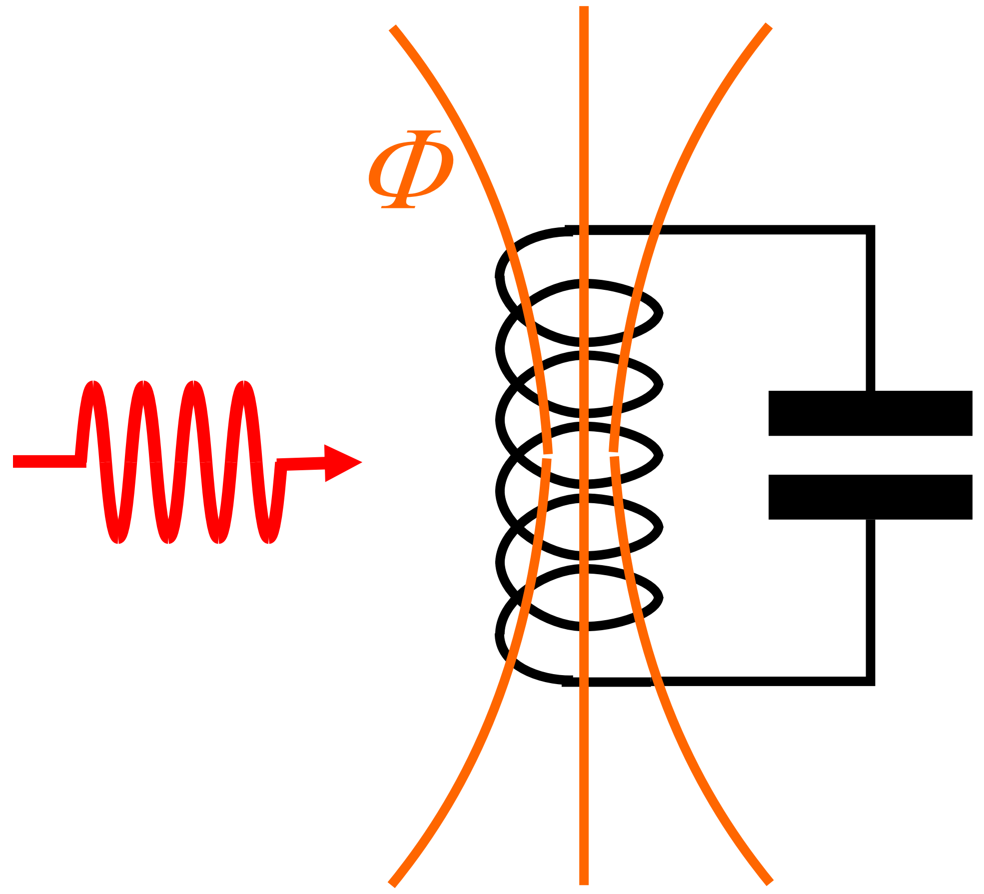

$$\omega_r = \frac{1}{\sqrt{LC}}$$

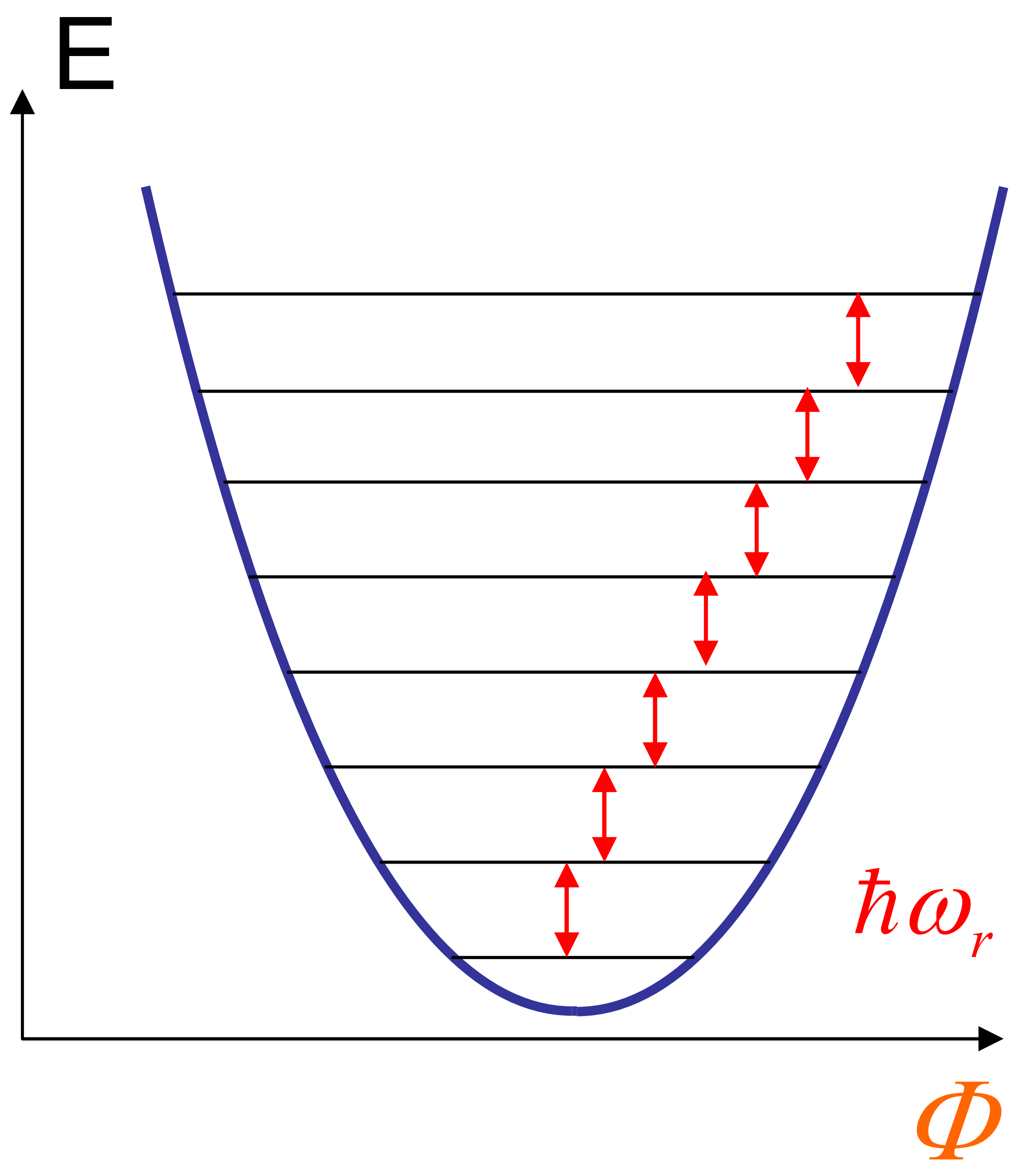
$$\left[\hat{\Phi}, \hat{Q}\right] = i\hbar$$

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} = -\frac{1}{2C}\frac{\partial^2}{\partial\phi^2} + \frac{\phi^2}{2L}$$

Courtesy of Michel Devoret

# LC oscillator is not enough

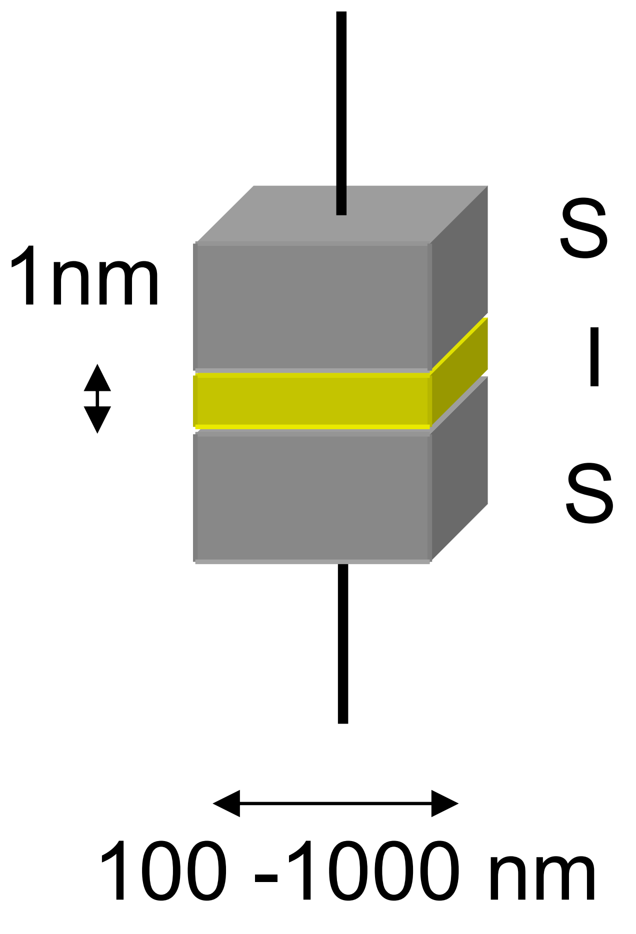


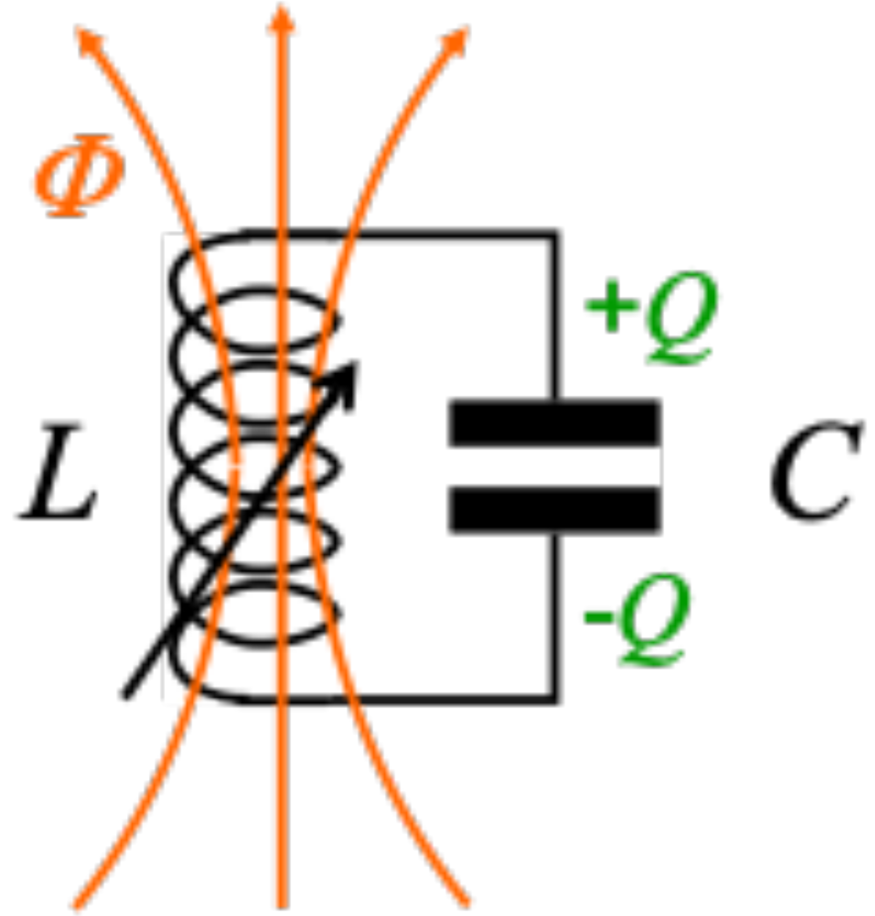

LC oscillator is a harmonic oscillator and we **cannot** selectively drive a transition between **two energy levels.**

Courtesy of Michel Devoret

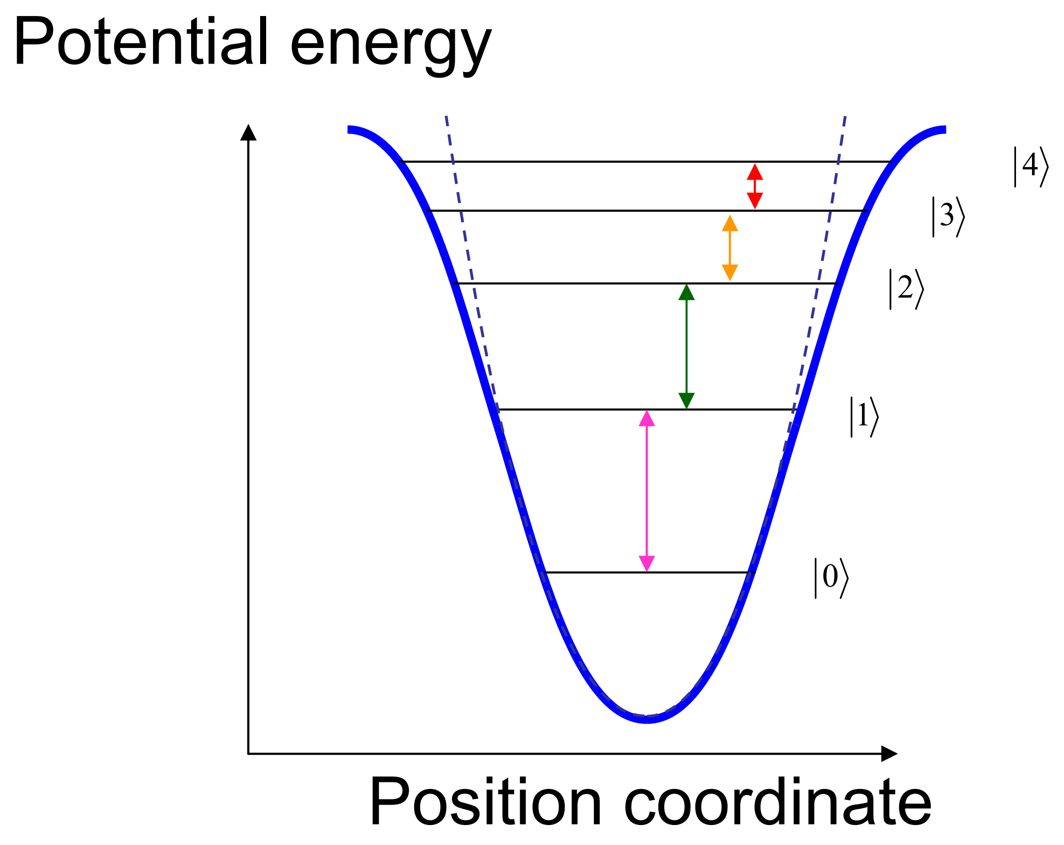
# Josephson Junctions for qubits

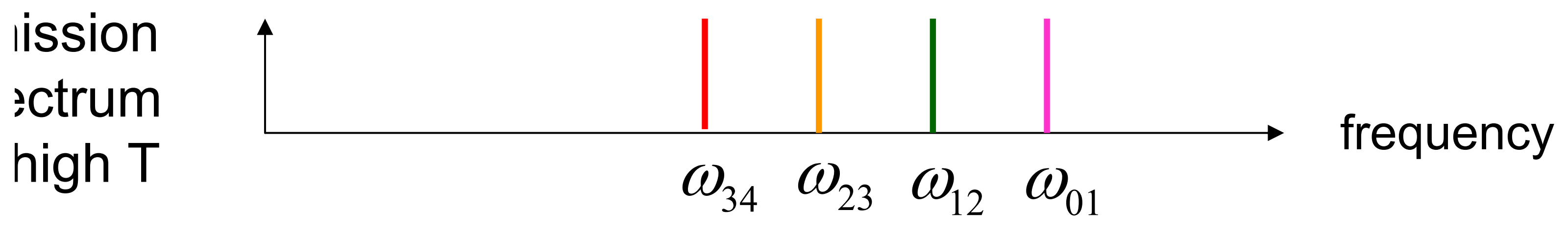



superconductor-
insulator-
superconductor
tunnel junction



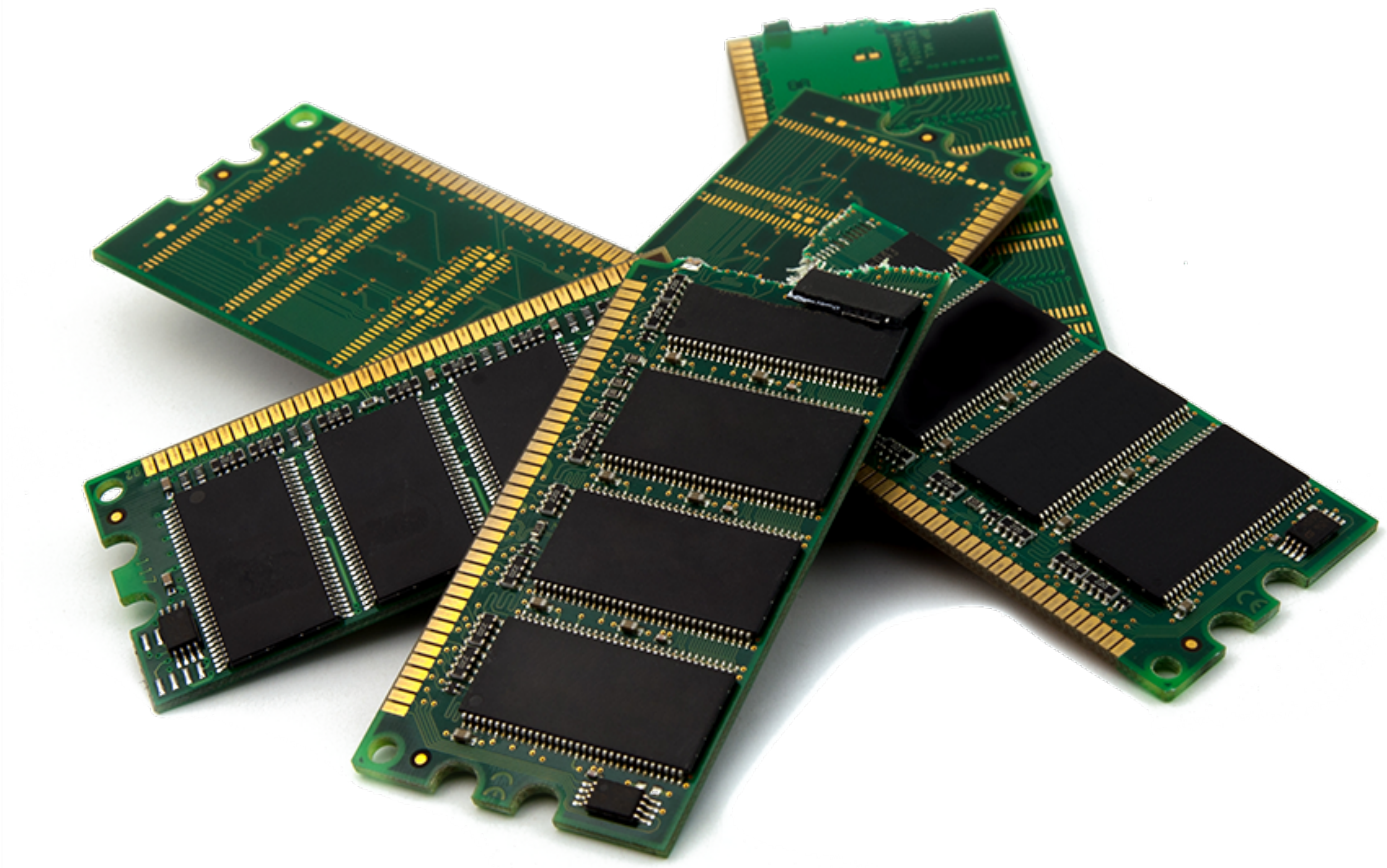

A nonlinear inductor with no dissipation

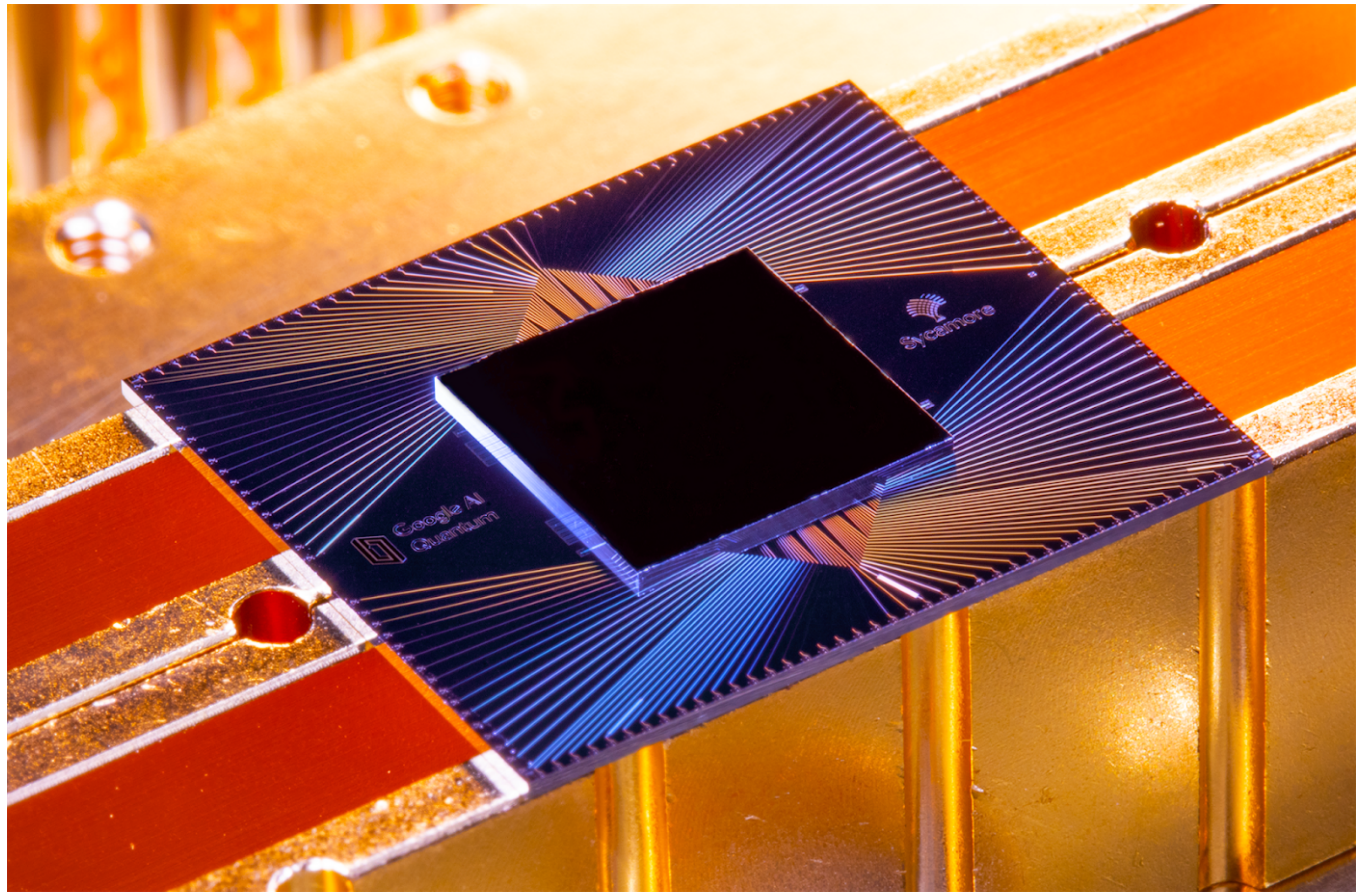



Courtesy of Michel Devoret

# Quantum hardware is (too) noisy





Quantum processor

$\sim 10^{-3} - 10^{-4}$ errors per bit per operation

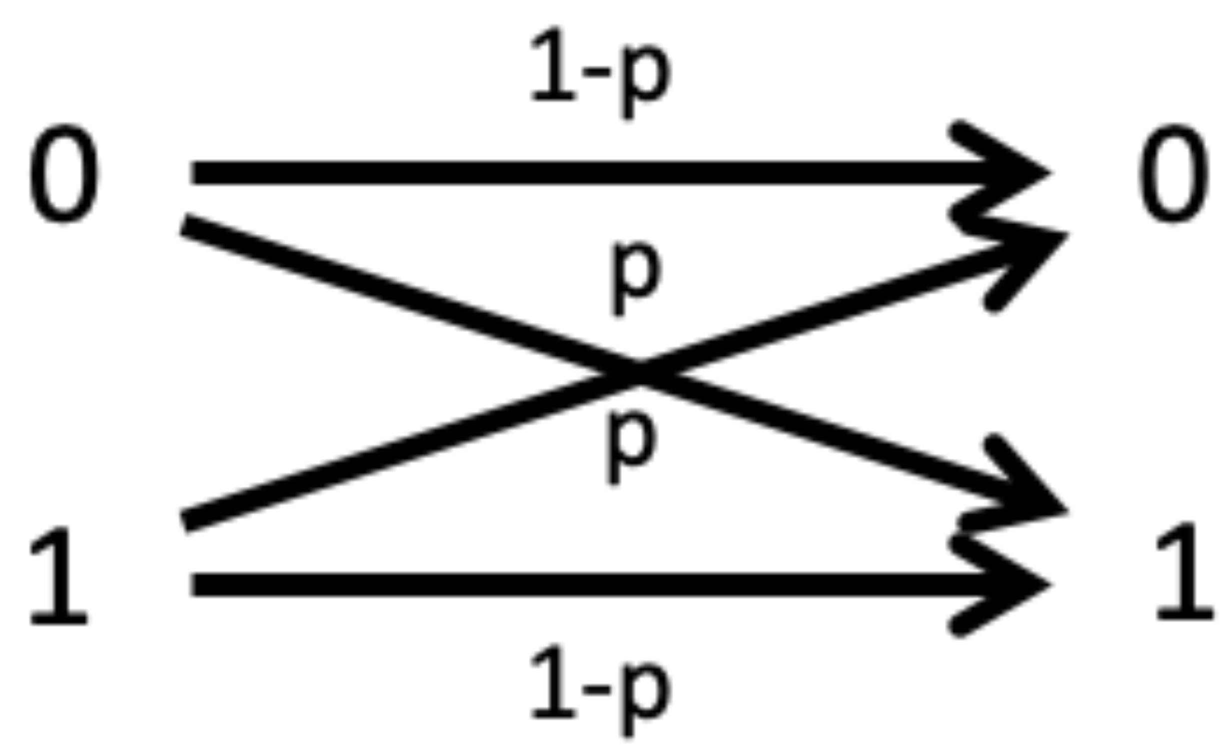
Classical RAM (Random Access Memory)

$\sim 10^{-25}$ errors per bit per operation

Large scale quantum computation requires $\sim 10^{-10} - 10^{-15}$

# Classical error correction

- Classical noise: bit-flip errors




- Idea of error correction: redundancy

$$0_L \to 000 \quad \text{and} \quad 1_L \to 111$$

- 1-bit errors tractable by majority vote

- Probability of failure: 2 or 3 errors. $p_L = 3p^2(1-p) + p^3$

- For p<1/2 $p_L < p$

# Quantum error correction: main issues

- How to detect errors without destroying quantum information? Measuring a qubit projects its state:

$$c_0|0\rangle + c_1|1\rangle \rightarrow 0 \text{ or } 1$$

- Errors are continuous: an error is not necessarily of the form $|\psi\rangle \rightarrow \sigma_x|\psi\rangle$.

  For instance, we can also have $|\psi\rangle \rightarrow e^{i\theta\sigma_x}|\psi\rangle \qquad \theta \in [0, 2\pi)$

- Errors are not only bit-flips or unitaires generated by $\sigma_x$ ? How to model errors?

Pauli operators: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

# 1- How to detect errors without destroying information?

**Objective**: protect any superposition $c_0|0\rangle + c_1|1\rangle$ without knowledge of coefficients.

**Tool**: repetition by entanglement $$c_0|0\rangle + c_1|1\rangle \rightarrow c_0|000\rangle + c_1|111\rangle$$

**Detection**: parity checks by measuring $Z_1 Z_2 = \sigma_z \otimes \sigma_z \otimes I$ and $Z_2 Z_3 = I \otimes \sigma_z \otimes \sigma_z$

# Quantum error correction: main issues

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$$c_0|0\rangle + c_1|1\rangle \rightarrow 0 \text{ or } 1$$

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# 2 and 3: How to model errors? What is error correction?

Language of open quantum systems:

State space:

$$|\psi\rangle \in \mathcal{H} \longrightarrow \rho \in \{\xi \in \mathcal{B}_1(\mathcal{H}) \mid \rho^\dagger = \rho, \rho \geq 0, \mathrm{Tr}(\rho) = 1\}$$

Particular case of a pure quantum state:

$$\rho = \Pi_{|\psi\rangle} = |\psi\rangle\langle\psi|$$

# 2 and 3: How to model errors? What is error correction?

Quantum noise: interaction with environment

System+environment $\in \mathcal{H}_s \otimes \mathcal{H}_{\text{env}}$

$$\mathcal{E}(\rho_s) = \boldsymbol{tr}_{\text{env}}\left[\boldsymbol{U}_\tau(\rho_s \otimes \rho_{\text{env}})\boldsymbol{U}_\tau^\dagger\right] = \sum_k \boldsymbol{E}_k \rho_s \boldsymbol{E}_k^\dagger \quad \text{with} \quad \sum_k \boldsymbol{E}_k^\dagger \boldsymbol{E}_k = \boldsymbol{I}.$$

Quantum error correction in a nutshell:

Code space: $\mathcal{H}_c \subset \mathcal{H}_s$

Like errors, the error correction (measurement and feedback) can be modelled as a quantum operation:

$$\rho \to \mathcal{R}(\rho) = \sum_k \boldsymbol{R}_k \rho \boldsymbol{R}_k^\dagger$$

This corrects an error channel $\rho \mapsto \mathcal{E}(\rho)$ if $\forall \rho_c \in \mathcal{H}_c, \quad \mathcal{R} \circ \mathcal{E}(\rho_c) = \rho_c.$

# Why does quantum error correction work?

Theorem: discretisation of error channels

If the operation $\mathscr{R}$ corrects the error channel $\mathscr{E}$, it corrects any other error channel $\mathscr{F}$ whose elements F_k are complex linear combinations of the elements E_k of $\mathscr{E}$ :

$$\mathcal{R} \circ \mathcal{E}(\rho_c) = \rho_c \qquad \Rightarrow \qquad \mathcal{R} \circ \mathcal{F}(\rho_c) = \rho_c.$$

Corollary: it suffices to correct the operations $\{I, \sigma_x, \sigma_z, \sigma_y = i\sigma_x\sigma_z\}$ to correct any single qubit errors.

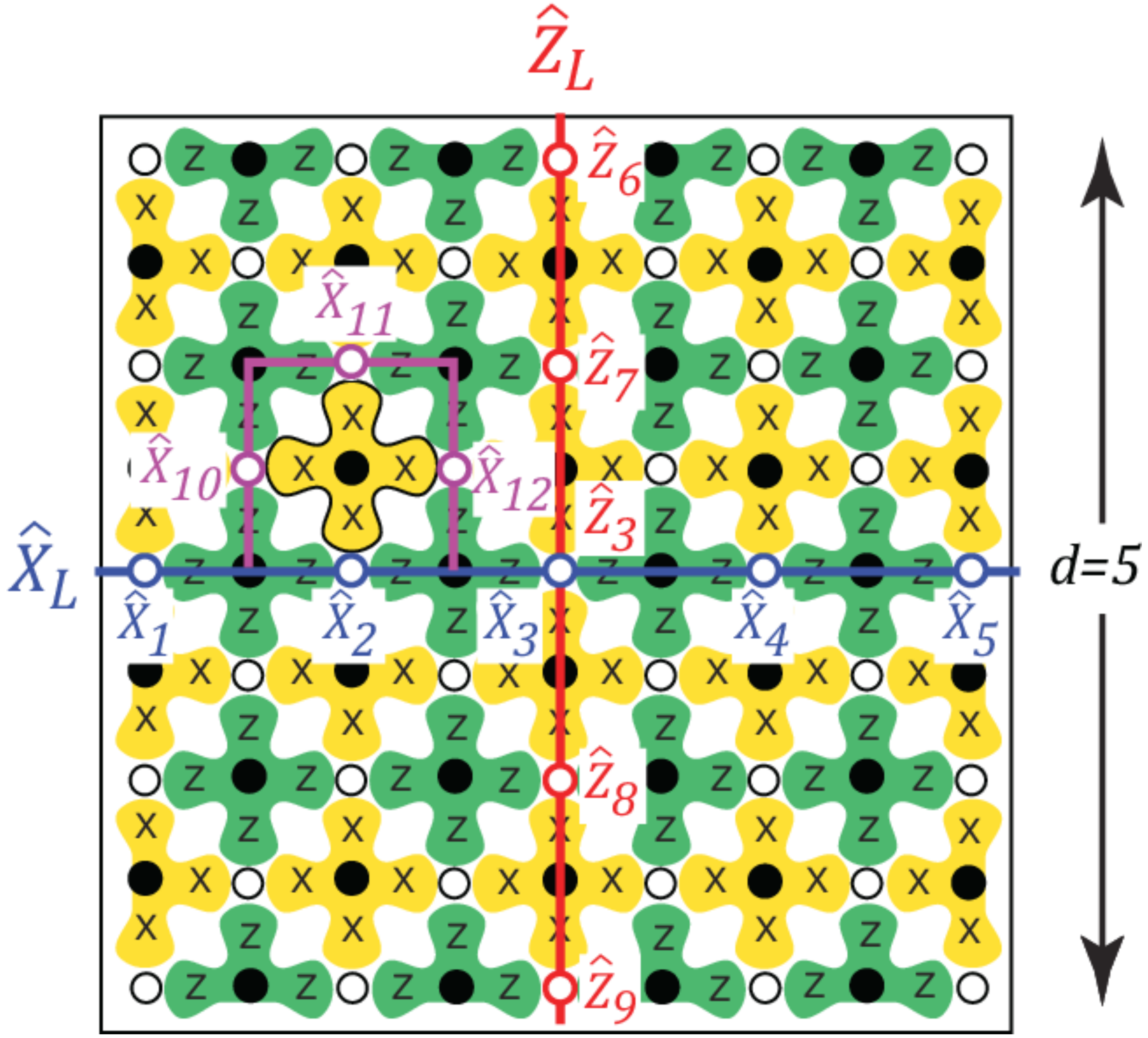
# From repetition code to surface code

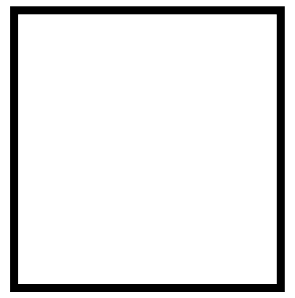
- Protect against I, X, Z and XZ errors

**Surface code**

$X_L =$ all $X$ along horizontal direction

$Z_L =$ all $Z$ along vertical direction

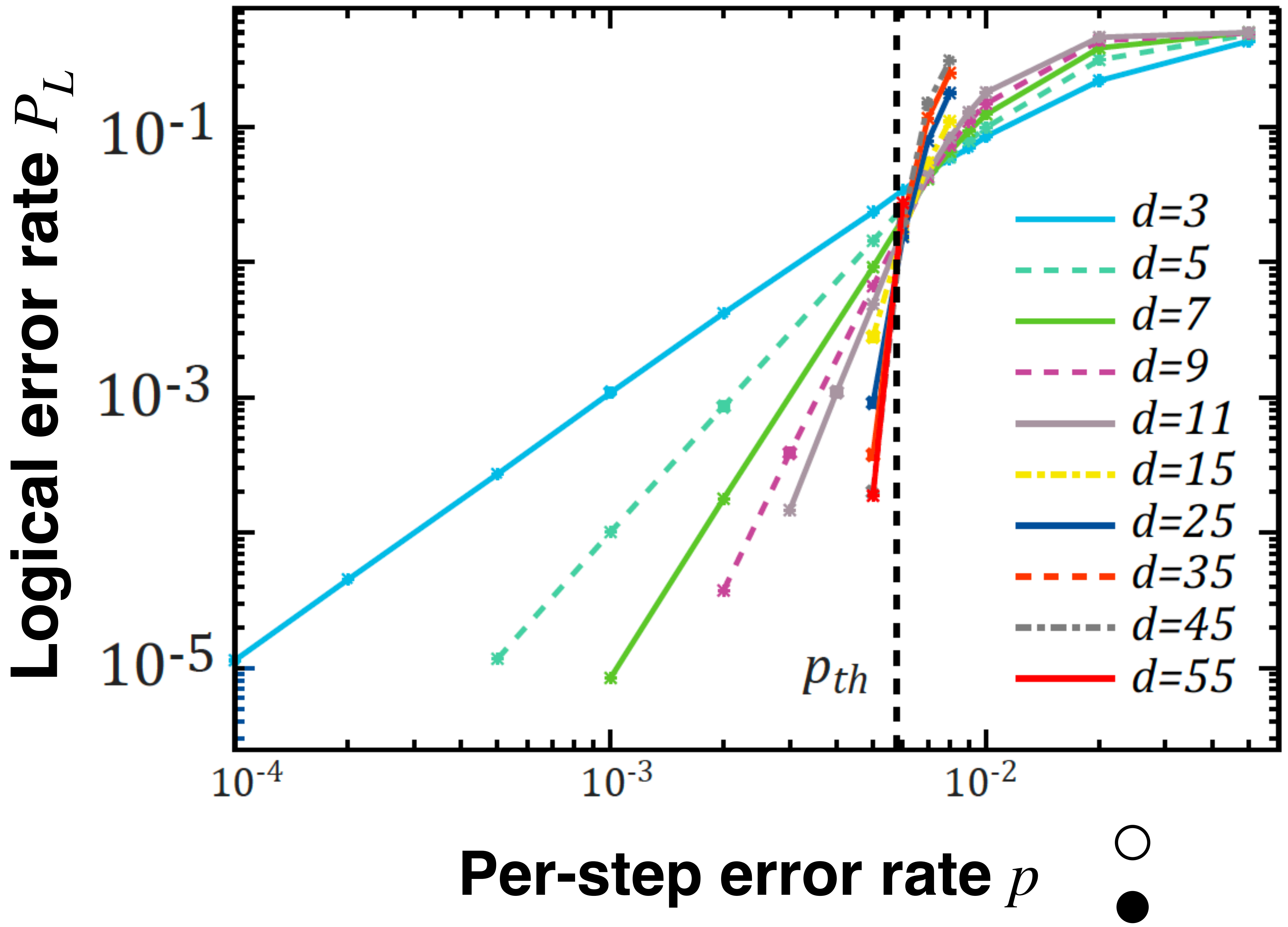



 } « logical » qubit

○ data qubit
● syndrome measurement qubit
} « physical » qubits

A. Fowler et al., Phys. Rev. A 86, 2012.

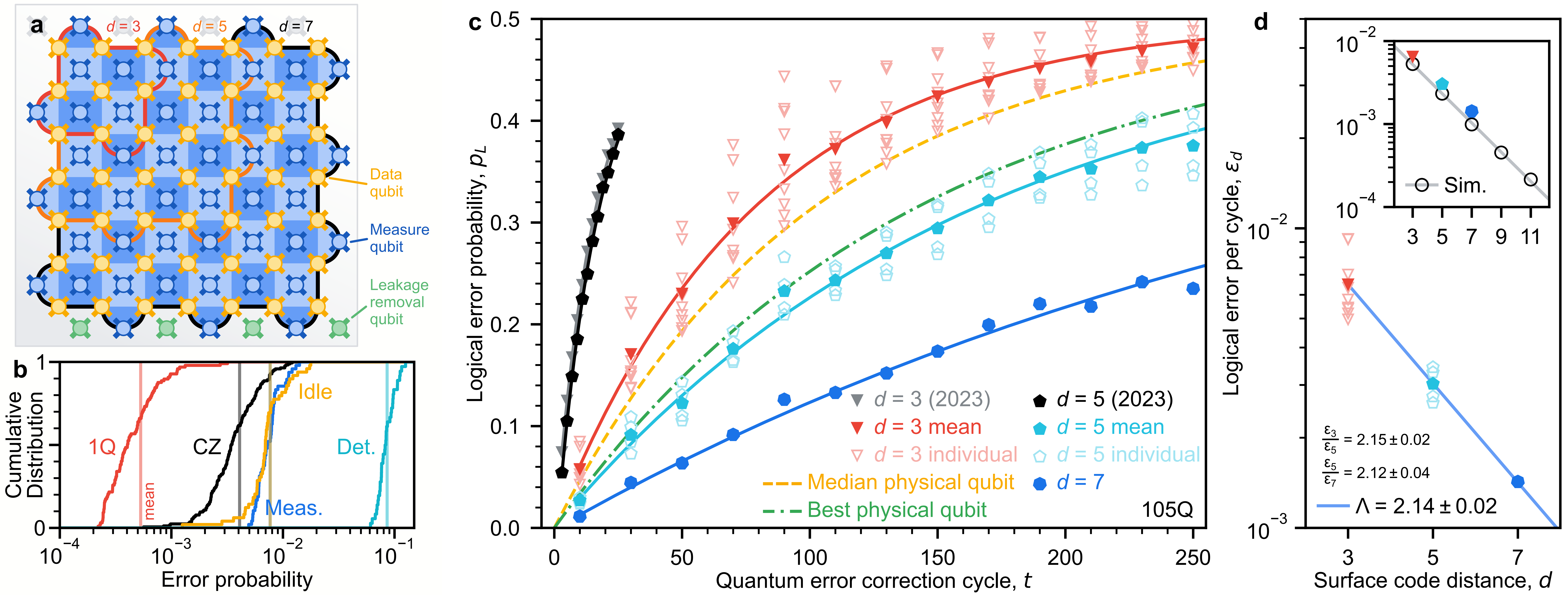
# Surface code: promise




$$p_L \propto \left(\frac{p}{p_{\mathrm{th}}}\right)^{\frac{d+1}{2}}$$

A. Fowler et al., Phys. Rev. A 86, 2012.

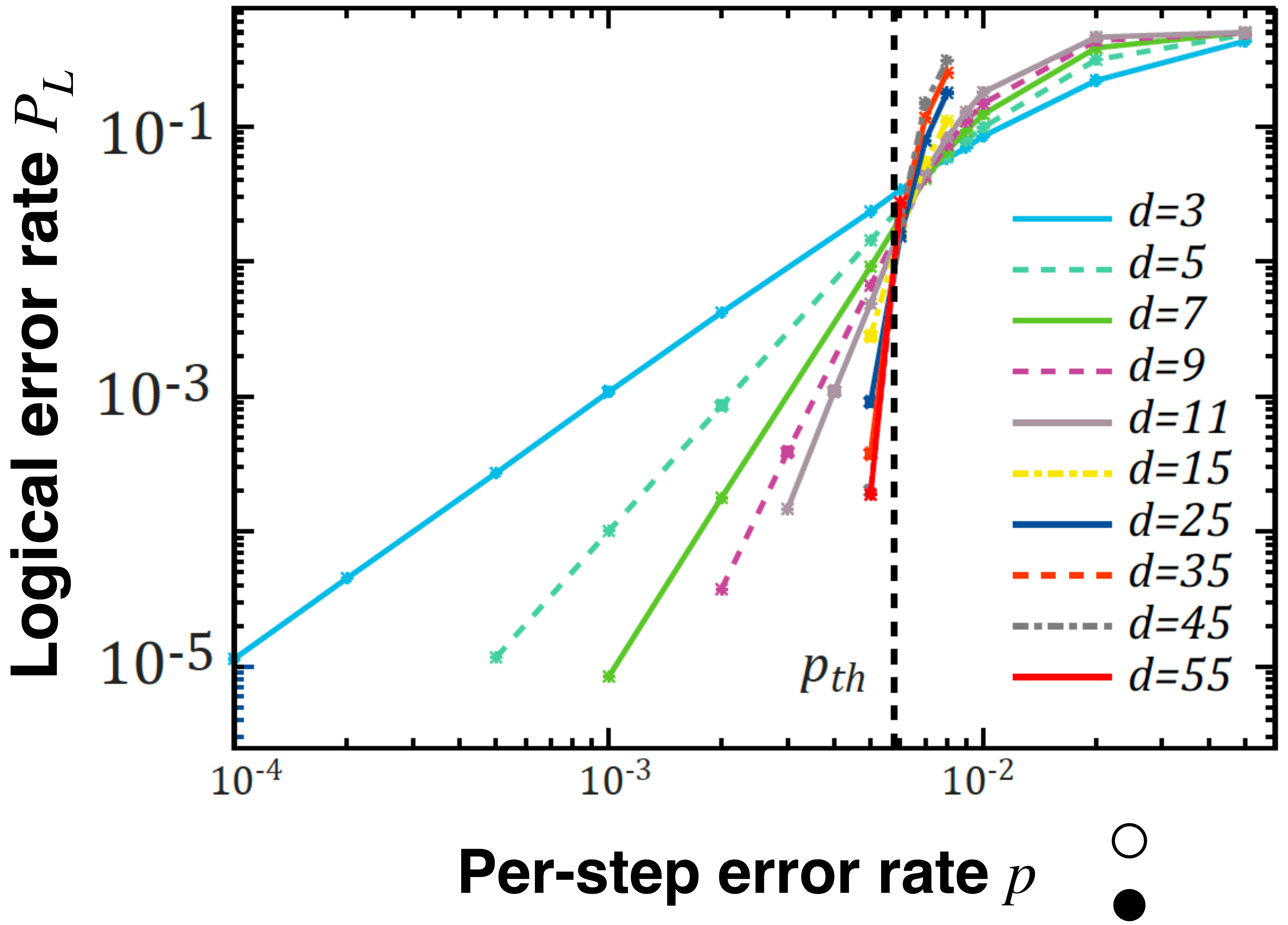
# State of progress



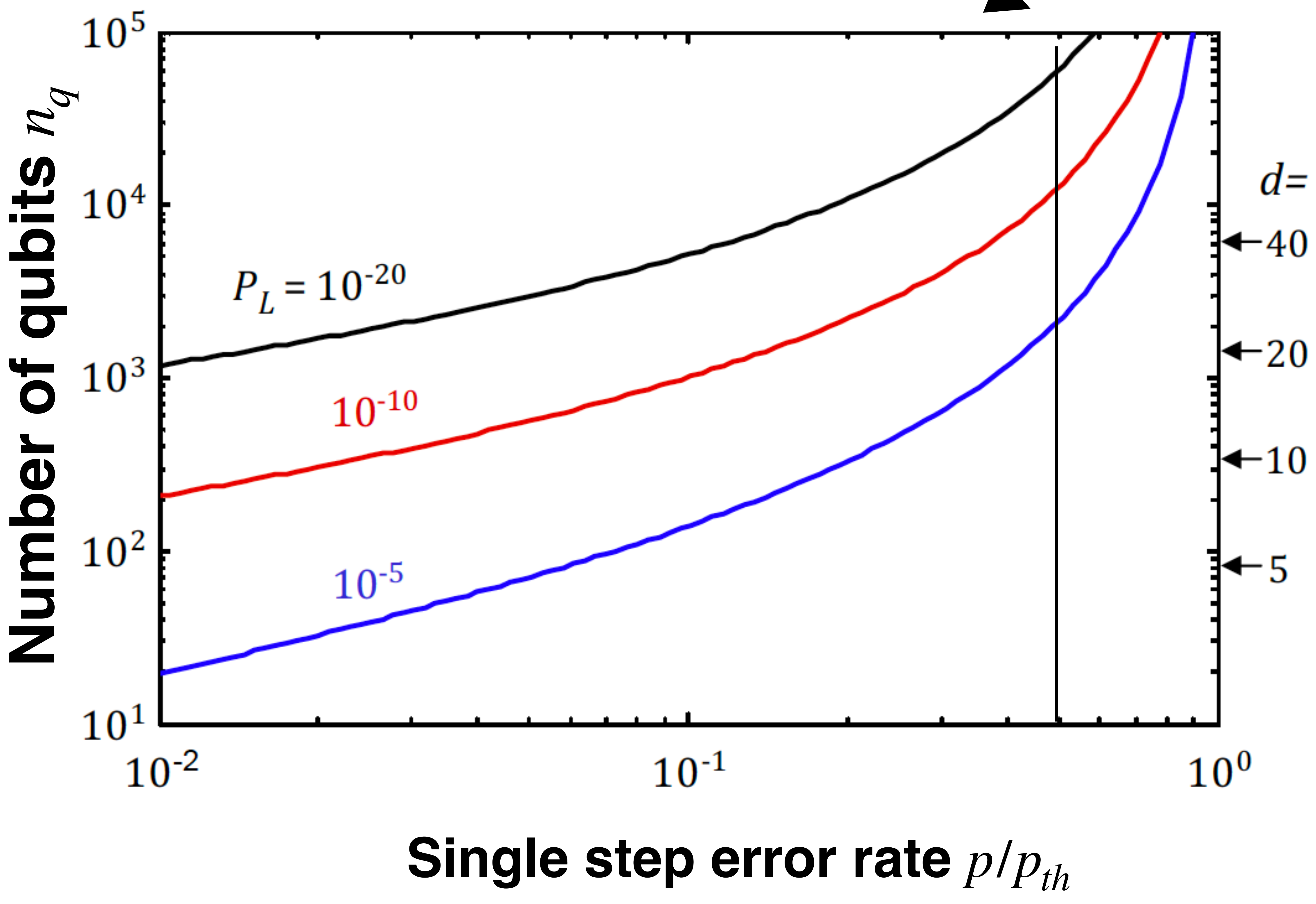

$$p_L \propto \left(\frac{p}{p_{\mathrm{th}}}\right)^{\frac{d+1}{2}} \qquad \Gamma := (p/p_{\mathrm{th}})$$

Google Quantum AI, Nature, 2025.

# State of progress



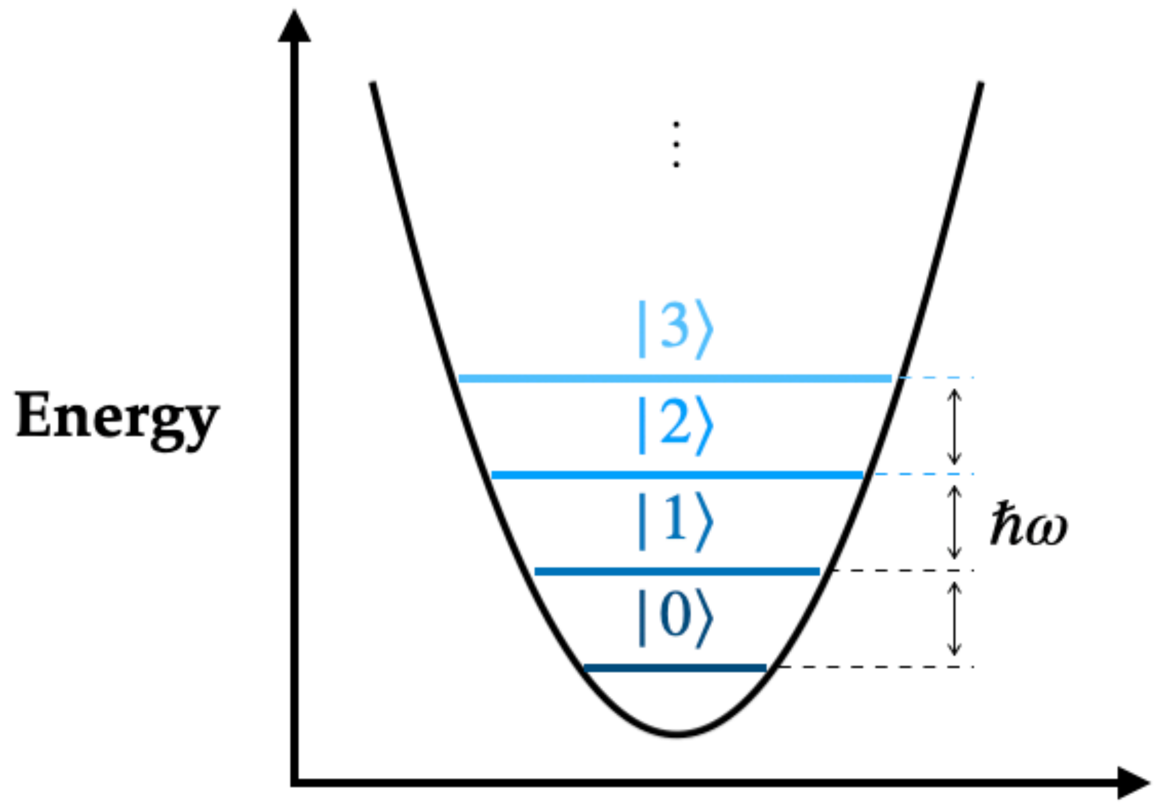

Google Quantum AI, ArXiv:2408.13687



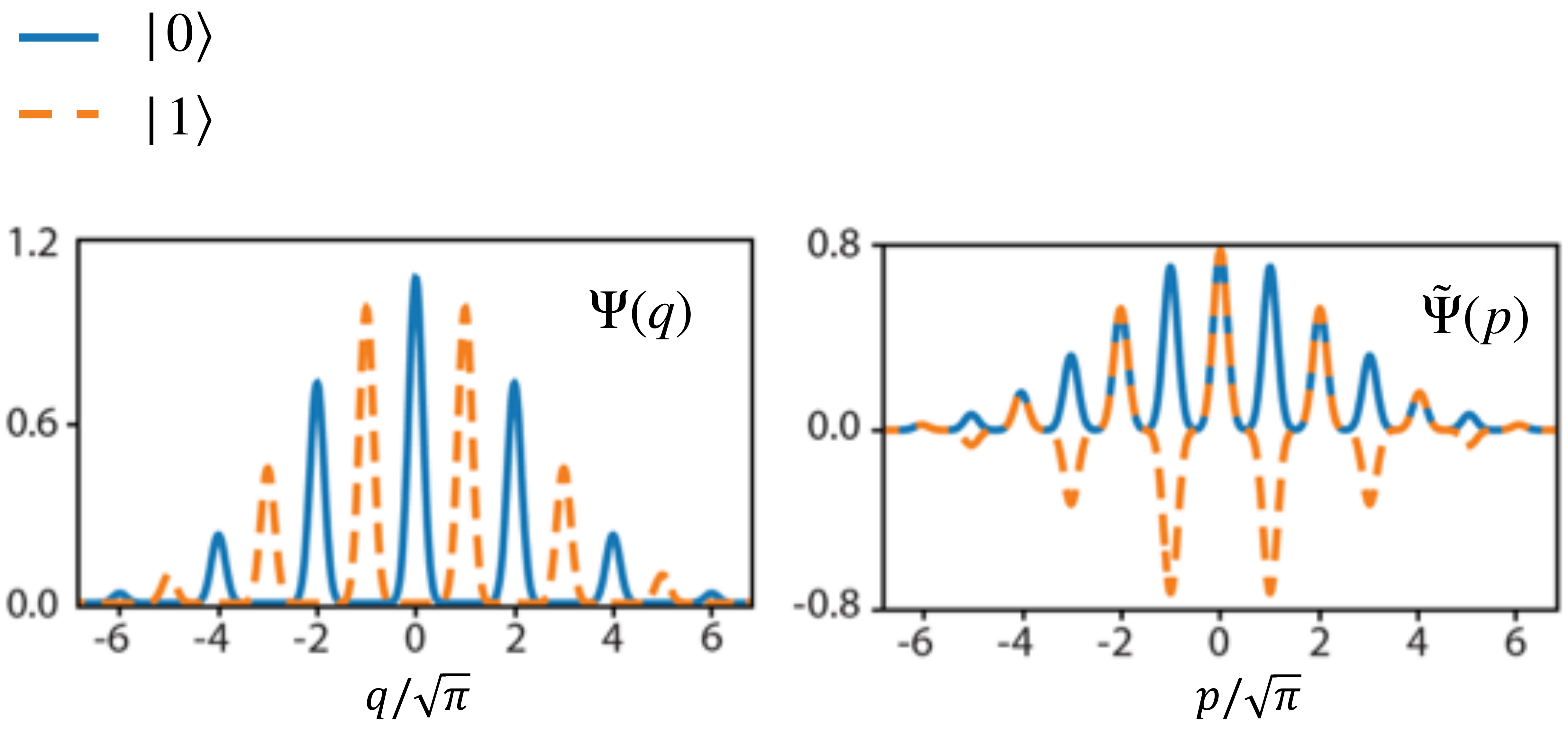

A. Fowler et al., Phys. Rev. A 86, 2012.

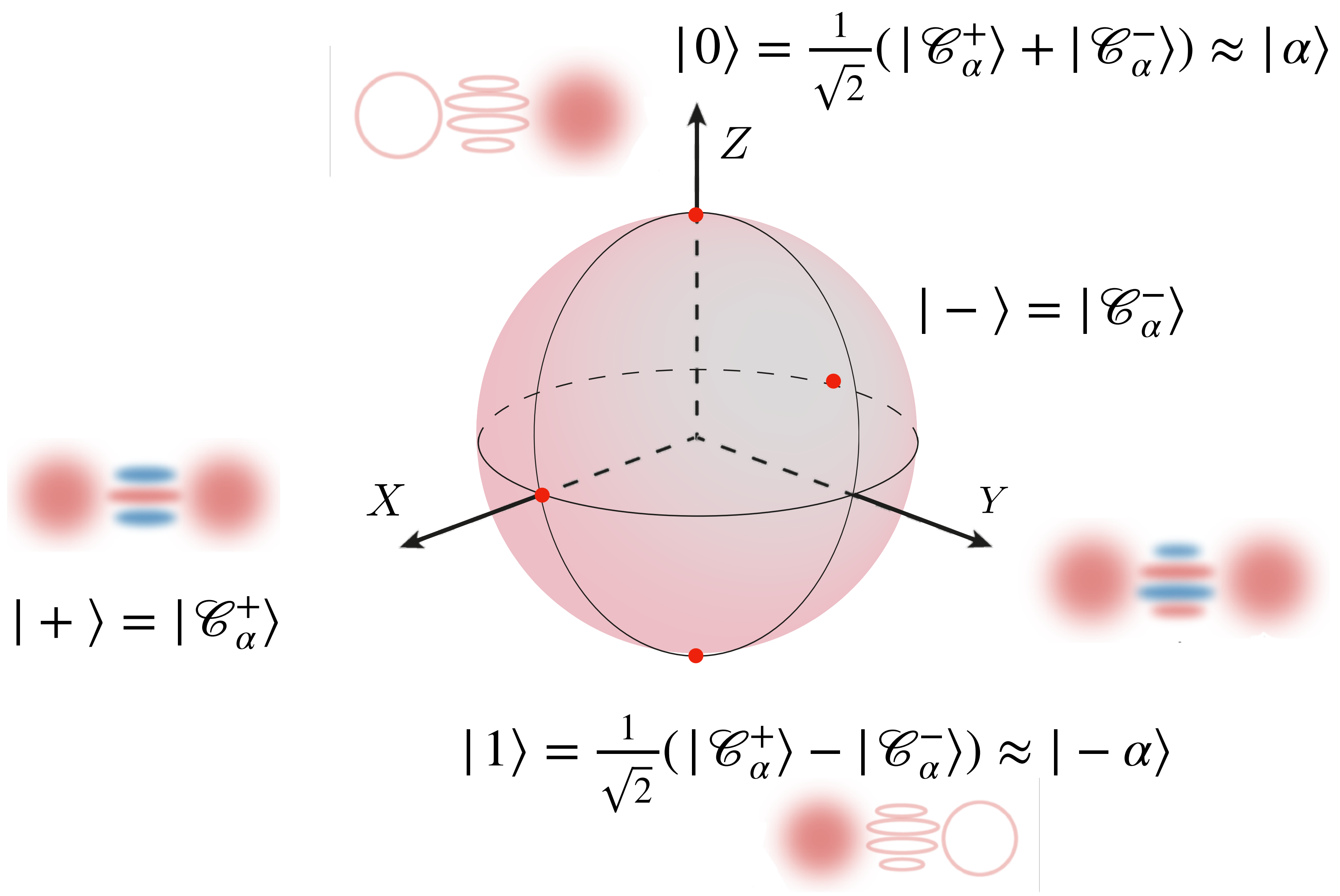
# Shortcut: QEC with bosonic codes

A. Joshi, K. Noh, Y. Gao,
Quantum Sci. Technol. 6 033001 (2021)




$$\mathscr{H} = \text{span}\{|n\rangle, n \in \mathbb{N}\}$$



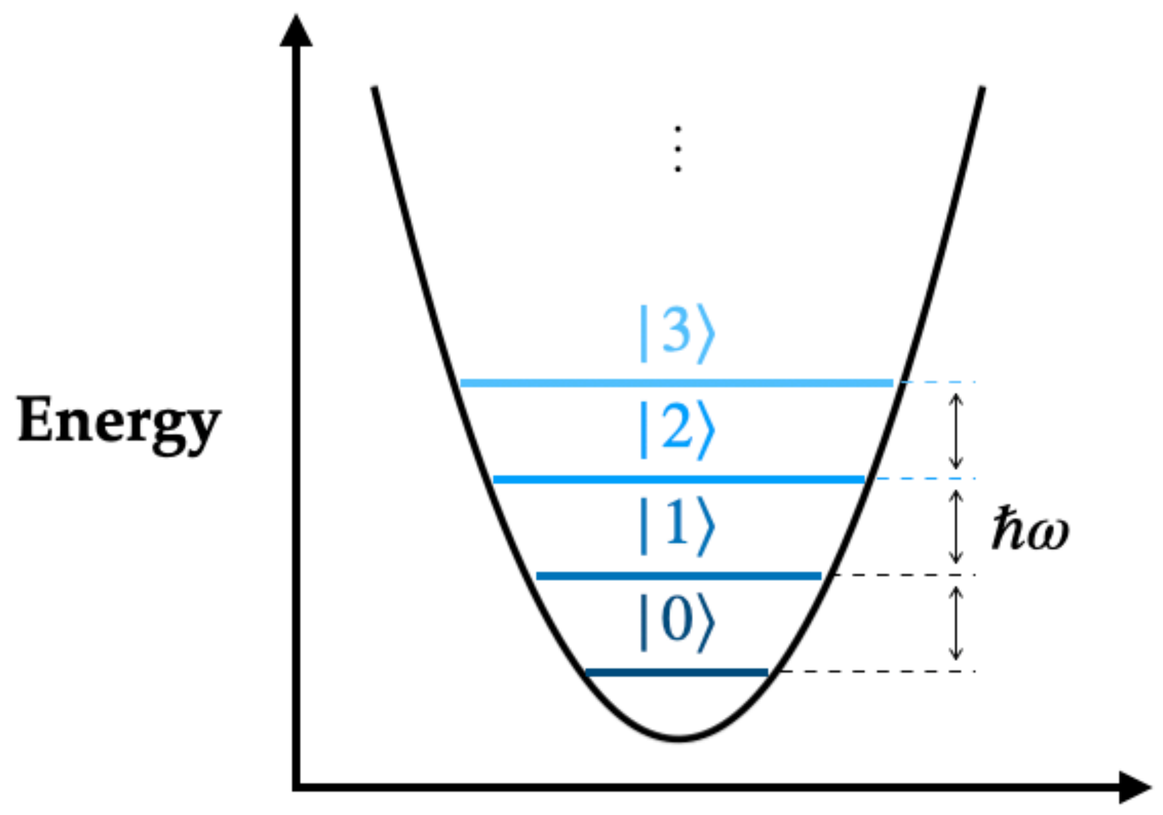




D. Gottesman, A. Kitaev, J. Preskill, Phys. Rev. A 64, 2001.

P.T. Cochrane et al., Phys. Rev. A 59, 1999.
Z. Leghtas et al., Phys. Rev. Lett. 111, 2013.

# Possible shortcuts: QEC with bosonic codes

Encode information in the infinite dimensional Hilbert space of a single quantum harmonic oscillator




Fock states: $\mathscr{H} = \text{span}\{|n\rangle, n \in \mathbb{N}\}$

$$|n\rangle = \psi_n(q) = (\frac{1}{\pi})^{1/4}\frac{1}{\sqrt{2^n n!}}e^{-q^2/2}H_n(q), \qquad H_n(q) = (-1)^n e^{q^2}\frac{d^n}{dq^n}e^{-q^2}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \textbf{and} \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$\hat{H} = -\frac{\hbar\omega}{2}\frac{\partial^2}{\partial q^2} + \frac{\hbar\omega}{2}q^2 = \hbar\omega(\hat{a}^\dagger\hat{a}+\frac{1}{2})$$

$$\hat{a} = \frac{1}{\sqrt{2}}(q + \frac{\partial}{\partial q}) \quad , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(q - \frac{\partial}{\partial q})$$

Coherent states: $\mathscr{H} = \text{span}\{|\alpha\rangle, \alpha \in \mathbb{C}\}$

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n\in\mathbb{N}}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = \frac{1}{\pi^{1/4}}e^{i\sqrt{2}q\Im(\alpha)}e^{-\frac{(q-\sqrt{2}\Re(\alpha))^2}{2}}$$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

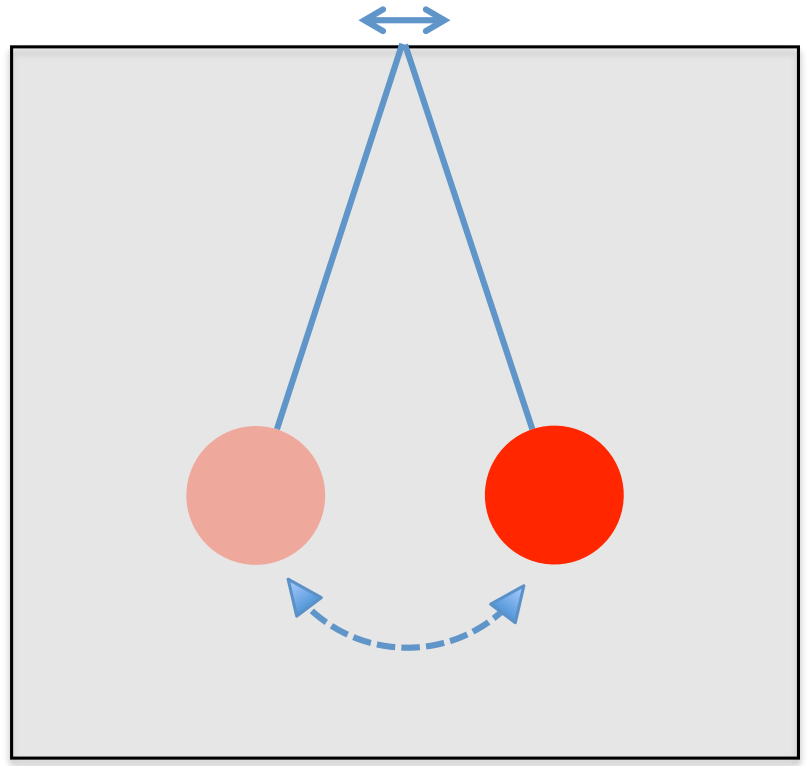
# Possible shortcuts: autonomous QEC with bosonic codes

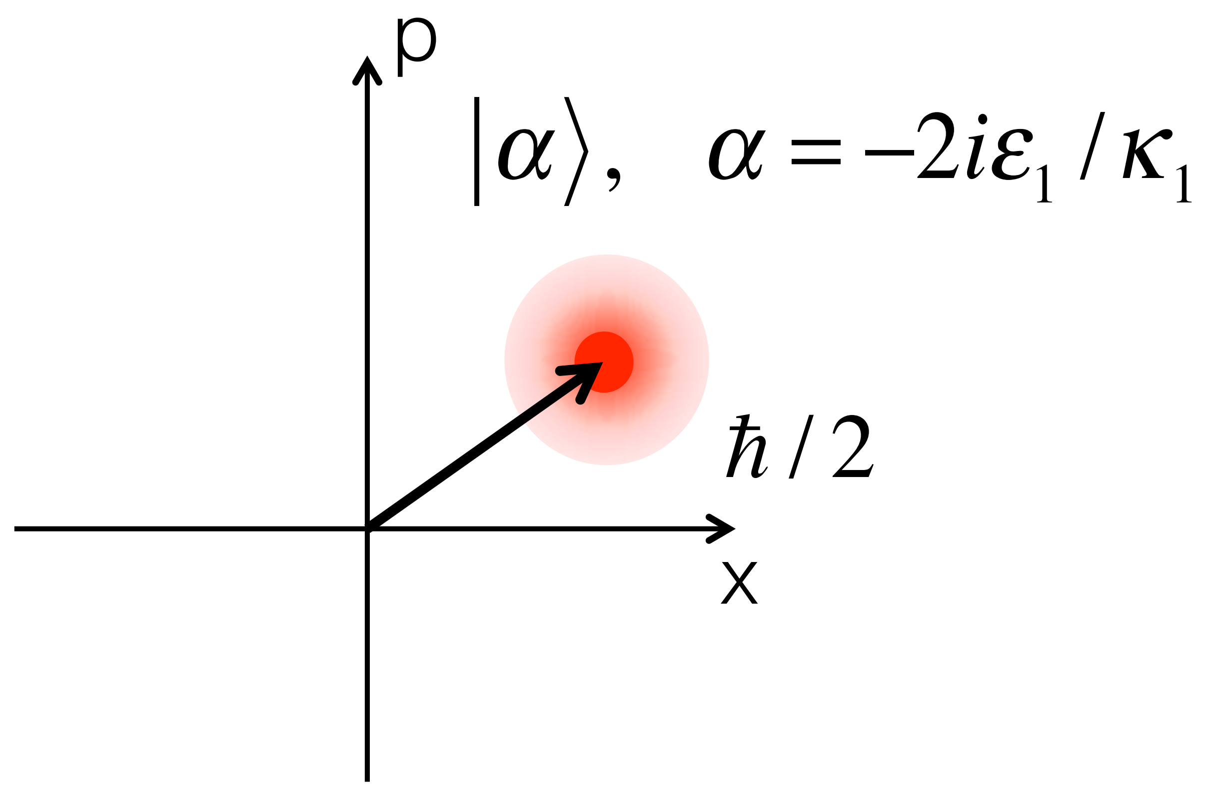
## « Single-photon » driven-damped harmonic oscillator

$$H = \epsilon_1^* \hat{a} + \epsilon_1 \hat{a}^\dagger \quad \text{and} \quad L = \sqrt{\kappa_1}\,\hat{a}$$

$$\frac{d\rho}{dt} = -i[H,\rho] + L\rho L^\dagger - \frac{1}{2}L^\dagger L\rho - \frac{1}{2}\rho L^\dagger L$$

$$\equiv \quad L = \sqrt{\kappa_1}(\hat{a} - \alpha)$$

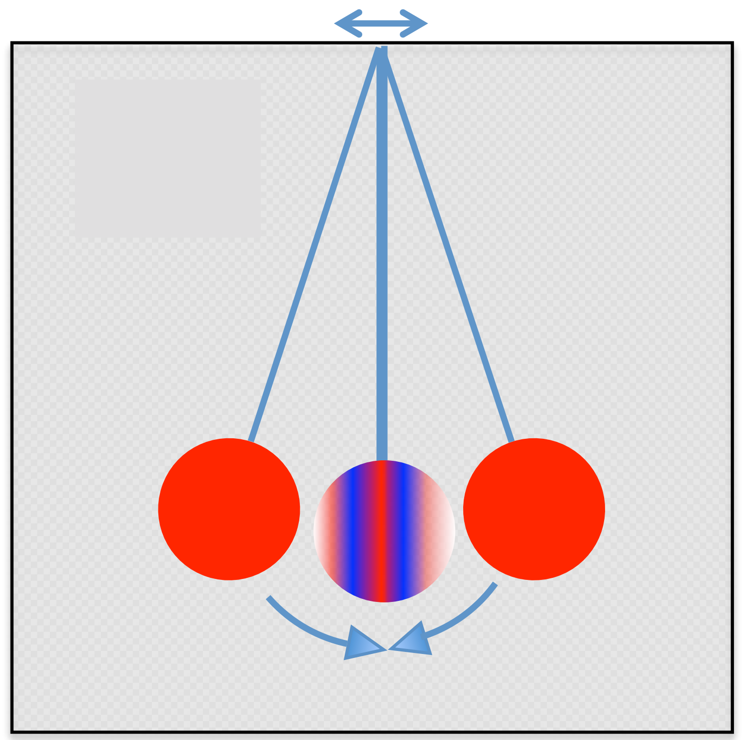


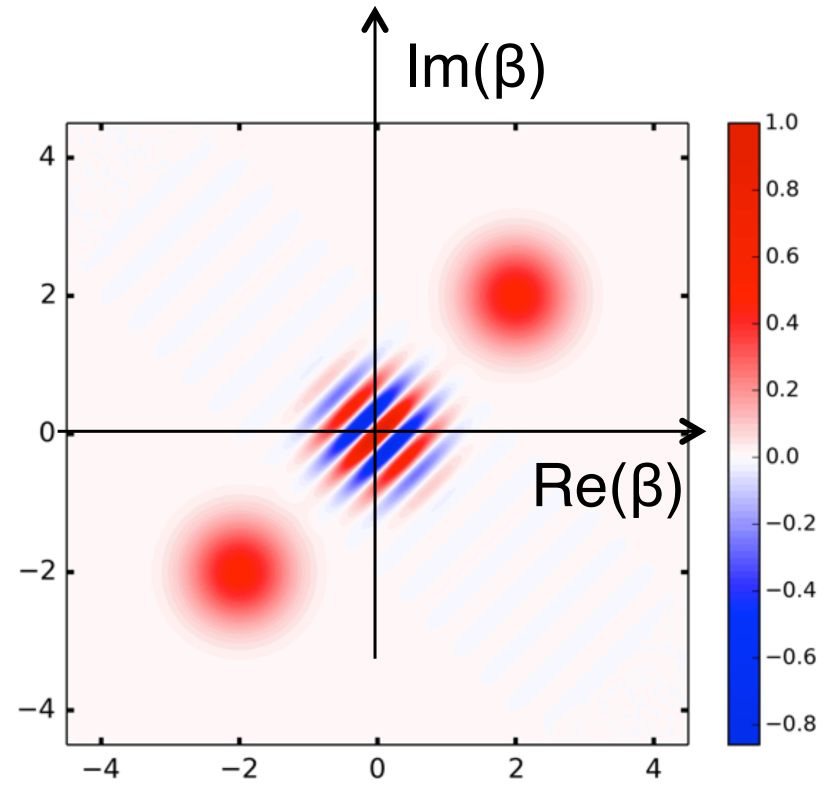



## « Two-photon » driven-damped harmonic oscillator

$$H = \epsilon_2^* \hat{a}^2 + \epsilon_2 \hat{a}^{\dagger 2} \quad \text{and} \quad L = \sqrt{\kappa_2}\,\hat{a}^2$$

$$\equiv \quad L = \sqrt{\kappa_2}(\hat{a}^2 - \alpha^2)$$





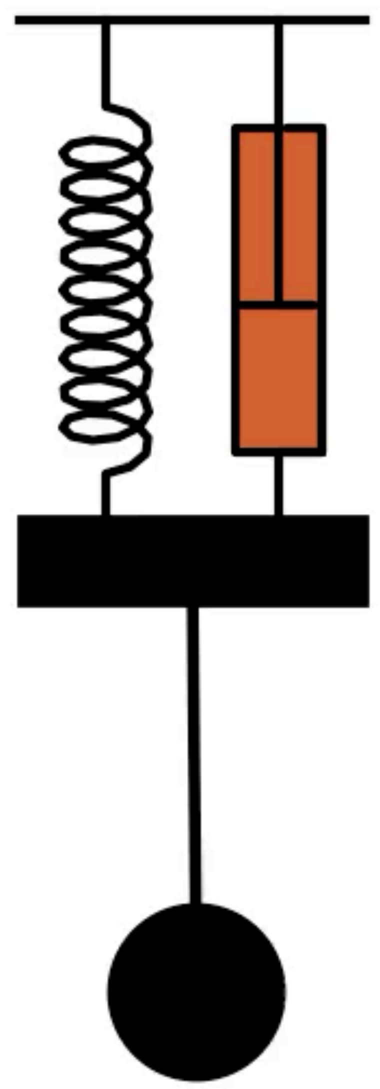
Wigner function W(β)

$$\{\,|\alpha\rangle, |-\alpha\rangle\}$$

$$\alpha = \pm\sqrt{-2i\varepsilon_2 / \kappa_2}$$

# Cat qubits: Autonomous QEC by dissipation engineering

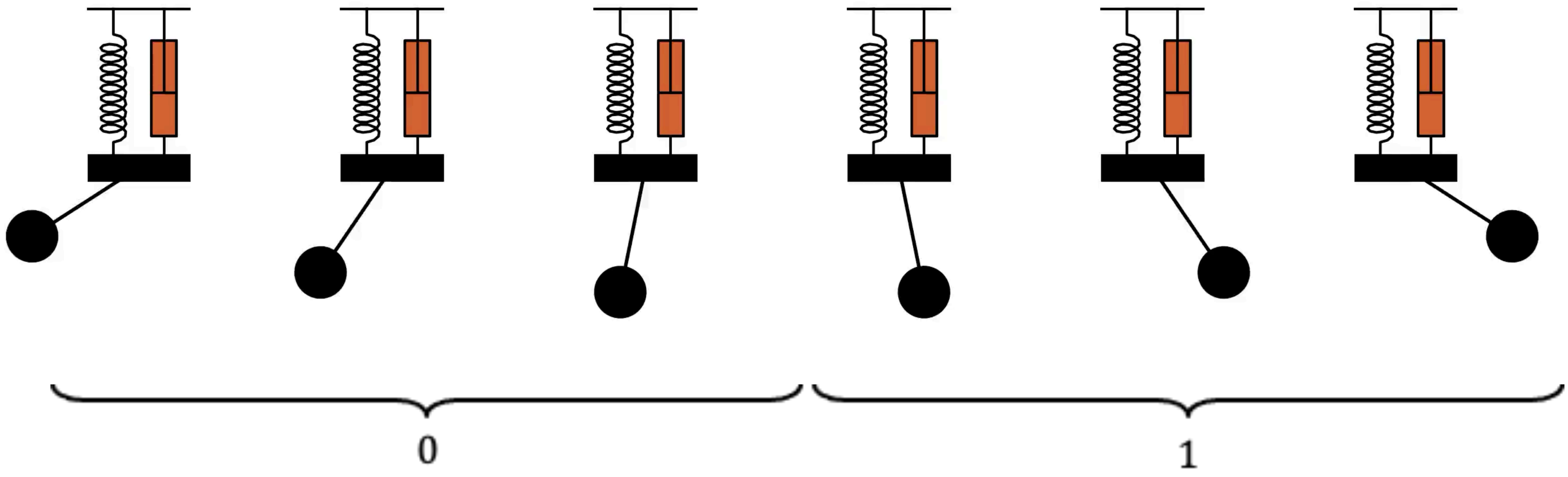
## A mechanical analog



Driven damped oscillator coupled to a pendulum

Courtesy of Raphaël Lescanne

# Mechanical analog: a bistable system

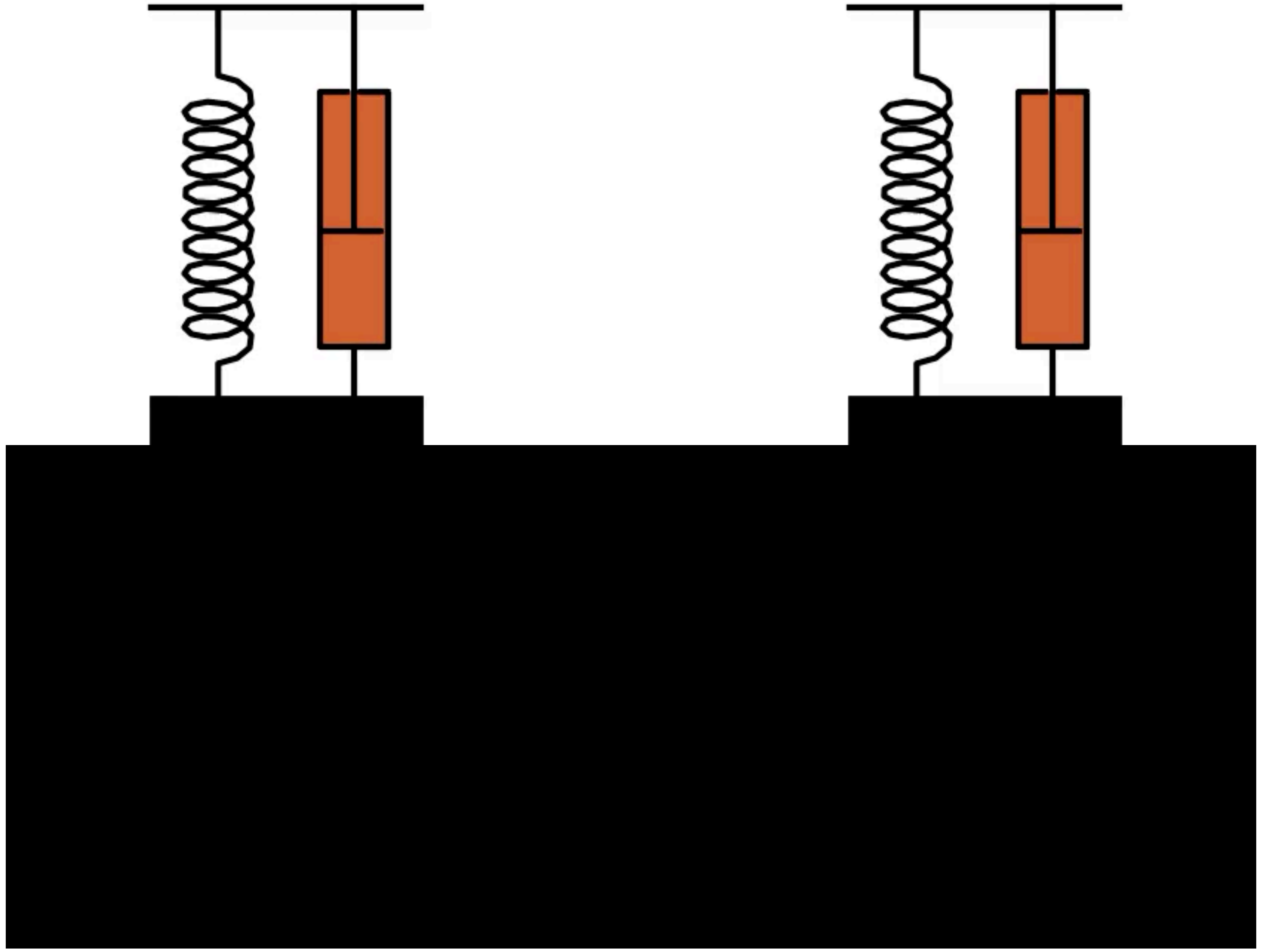



There are two steady states in which we encode information

Courtesy of Raphaël Lescanne

# Mechanical analog

**Stabilization regardless of the state**



Neither the **drive** not the **dissipation** can **distinguish** between 0 and 1

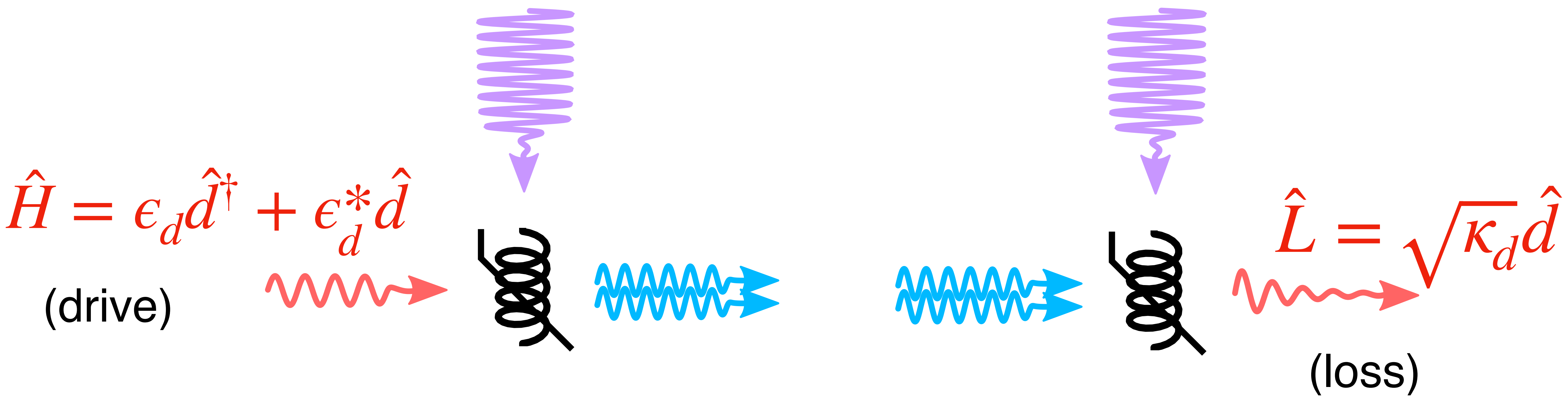
**Important to preserve quantum coherence**

# Quantum version

Parametric pumping for 2-photon coupling

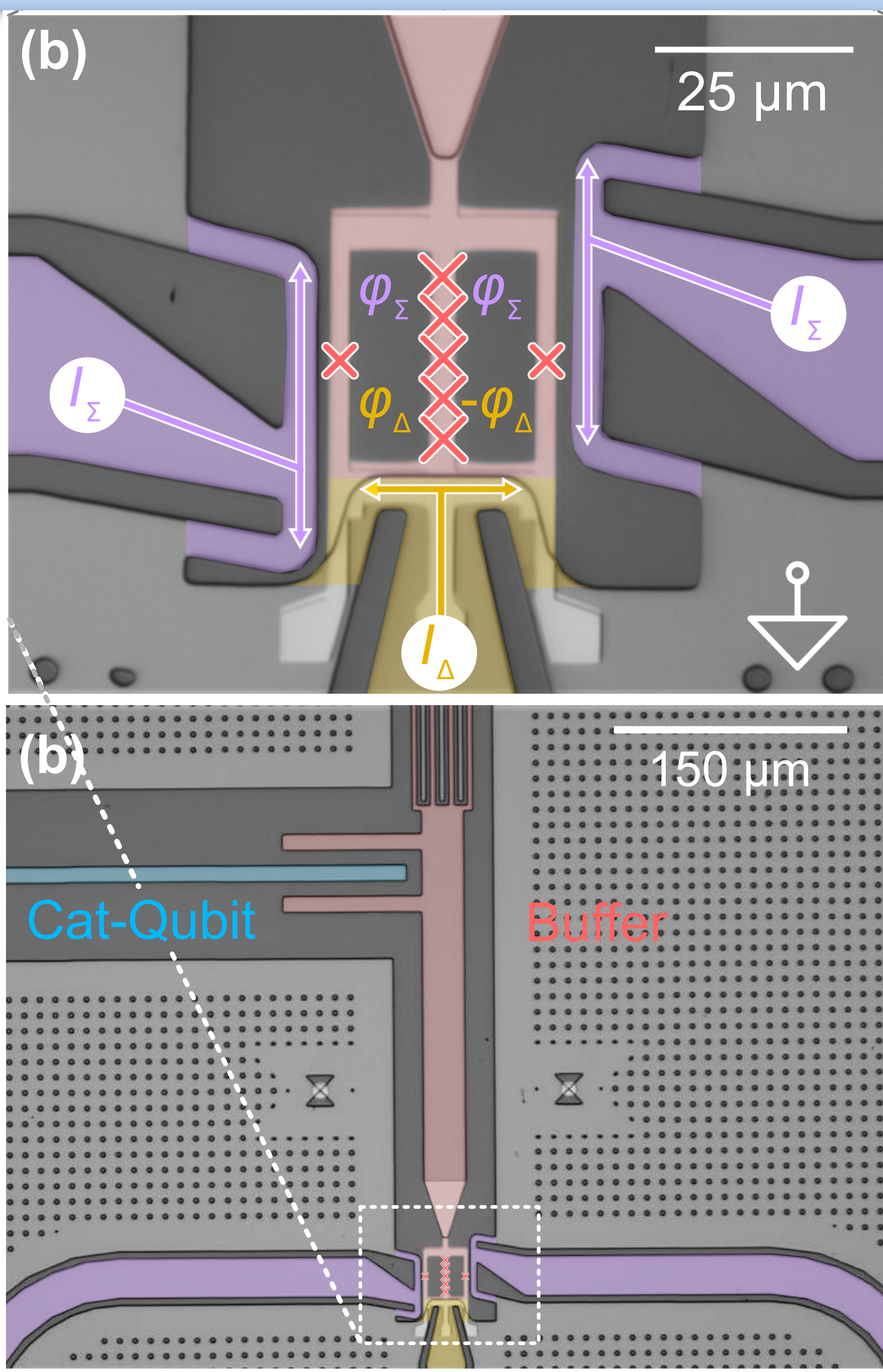
$$\hat{H} = g_2 \hat{a}^{\dagger 2}\hat{d} + g_2 \hat{a}^2 \hat{d}^{\dagger}$$

$$\boxed{\omega_p = 2\omega_a - \omega_d}$$



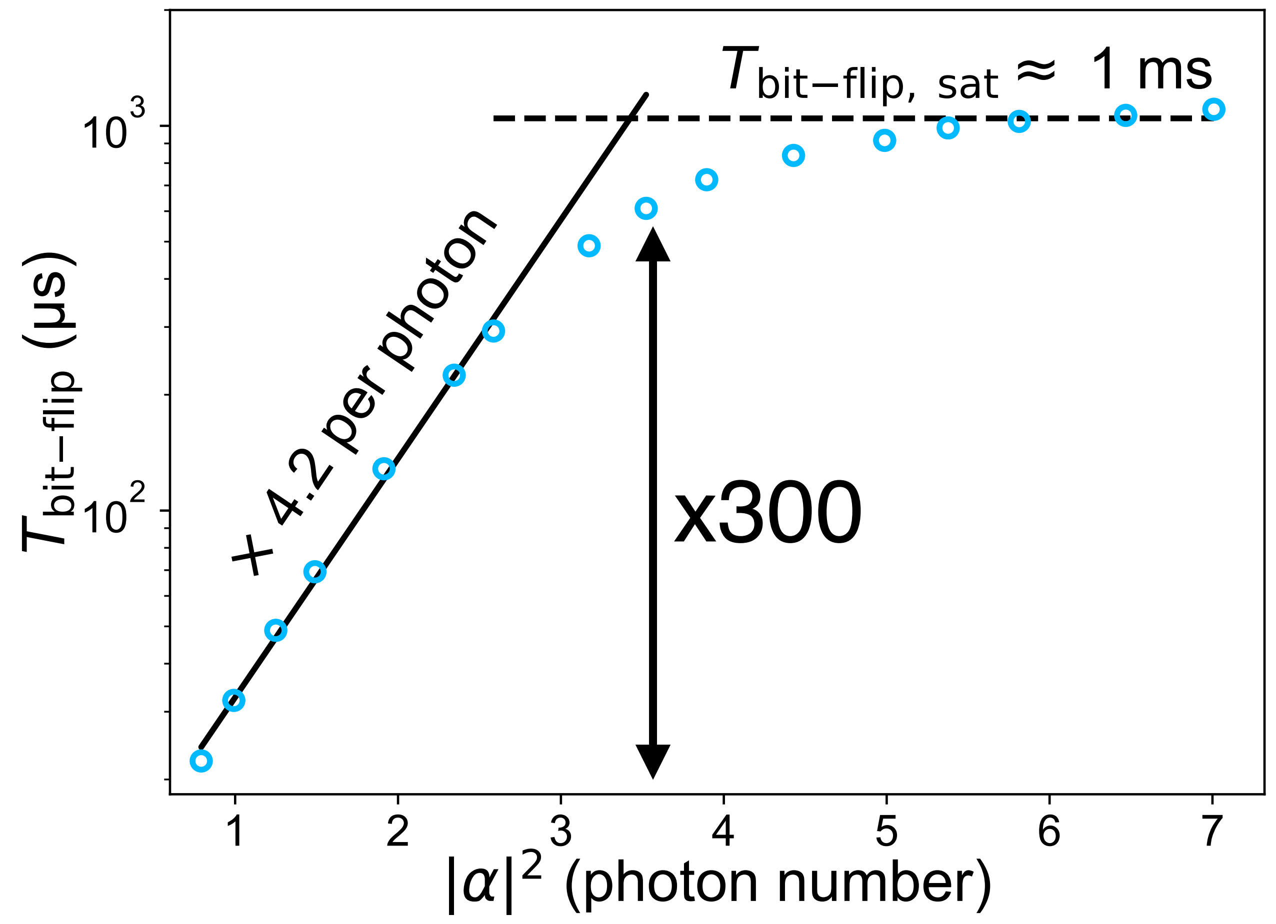

$$\boxed{\hat{H}_{eff} = \epsilon_2^* \hat{a}^2 + \epsilon_2 \hat{a}^{\dagger 2}}$$
(two-photon drive)

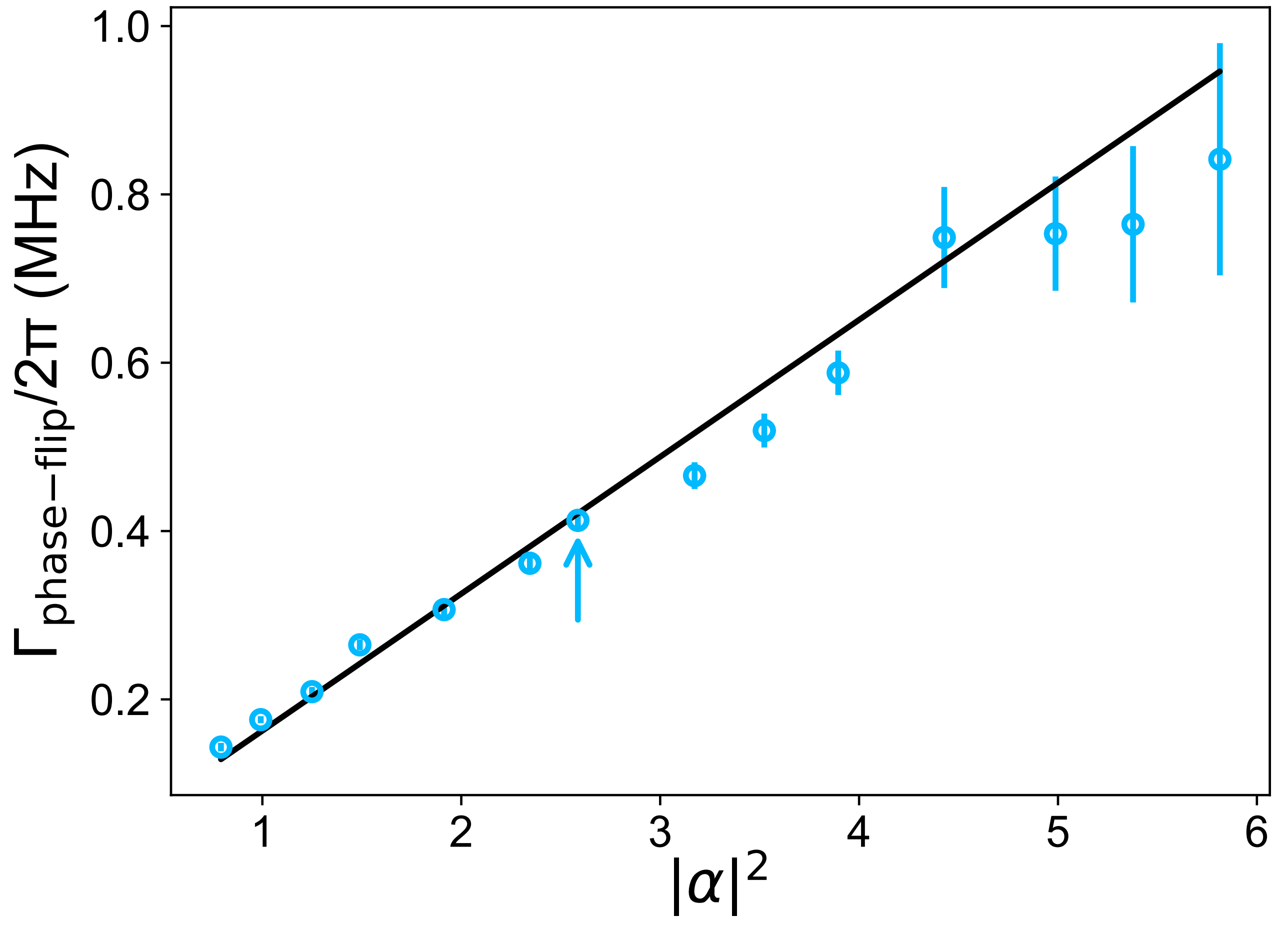
$$\boxed{\hat{L}_{eff} = \sqrt{\kappa_2}\hat{a}^2}$$
(two-photon loss)



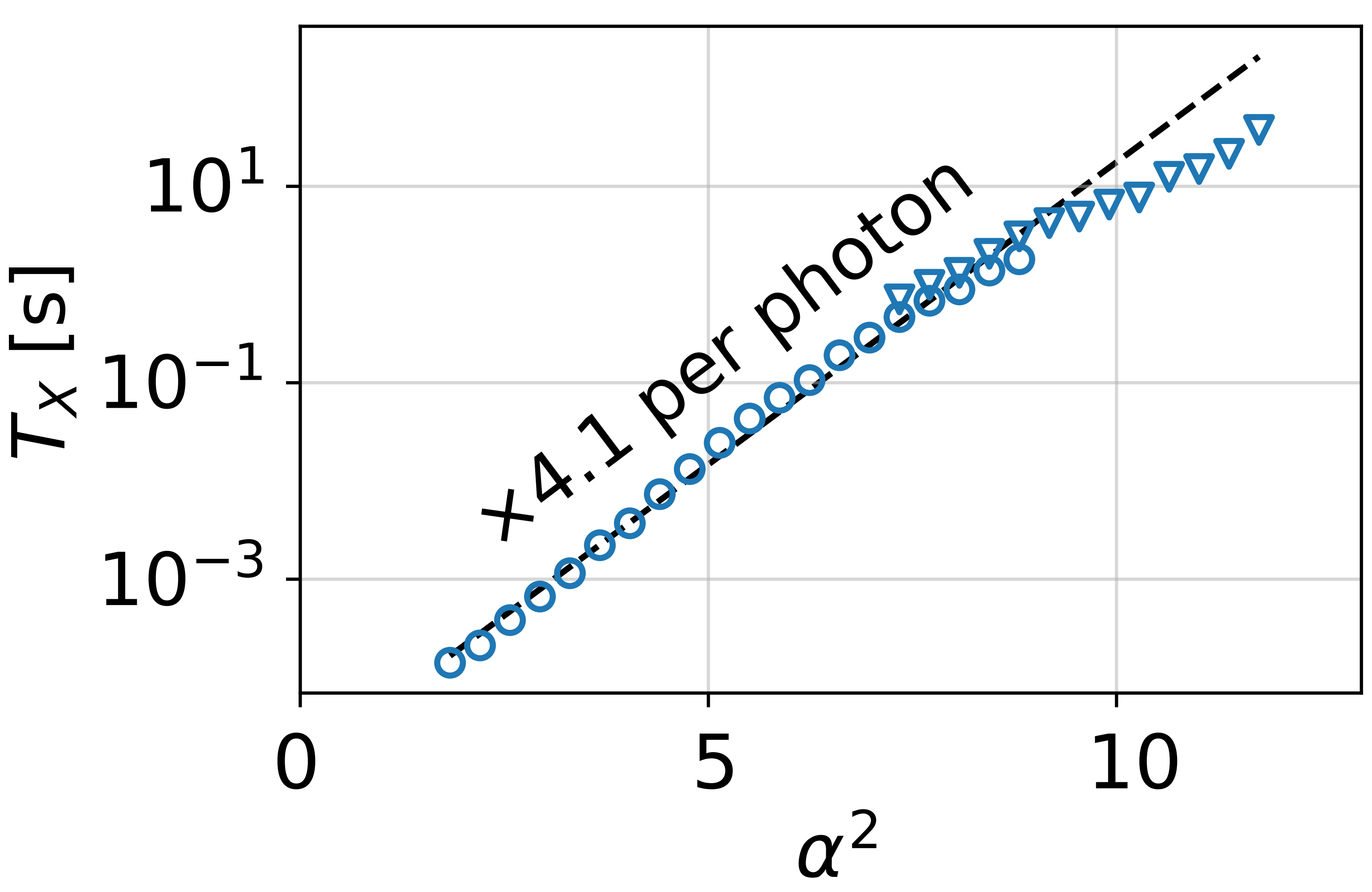

R. Lescanne et al., Nature Physics 2020

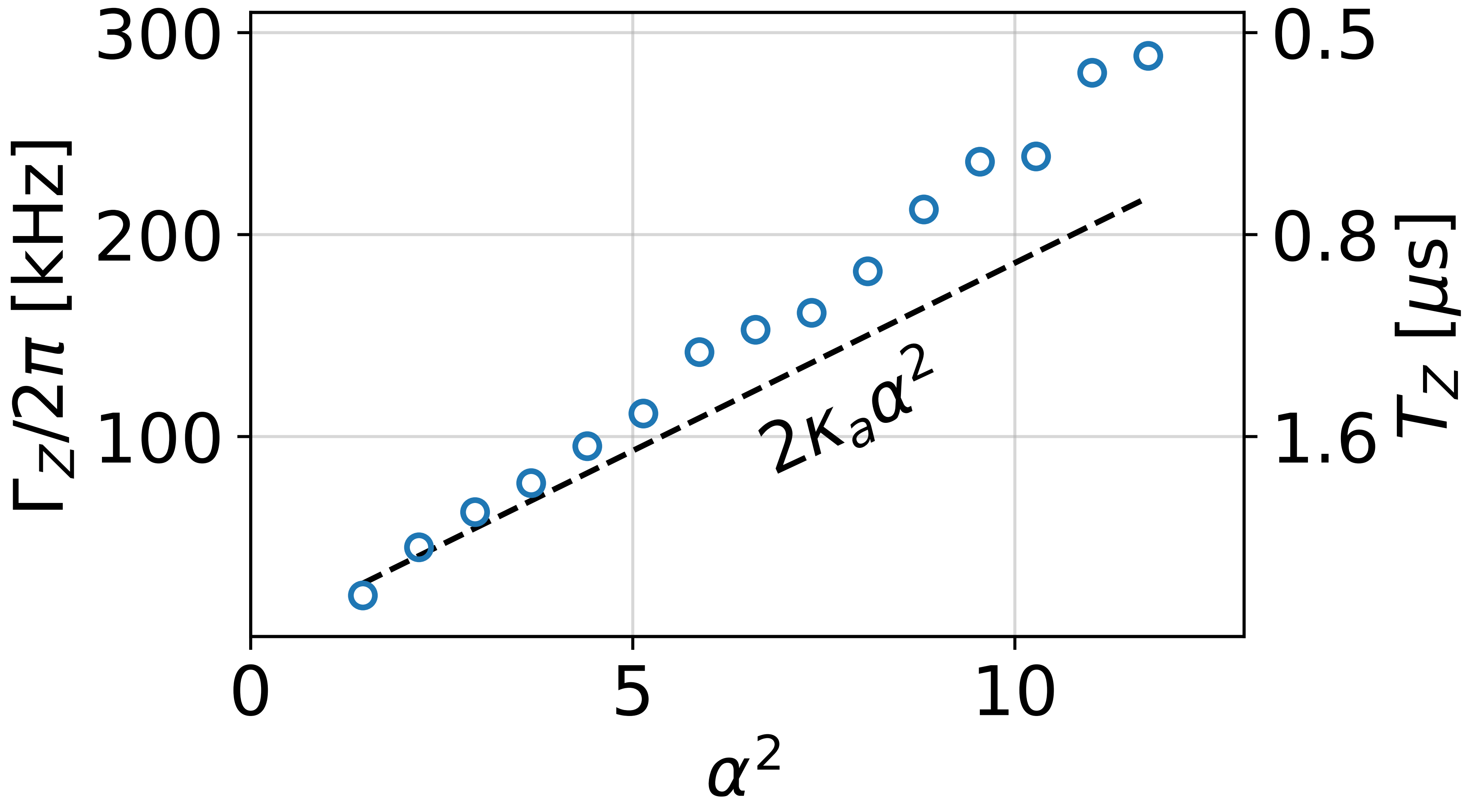
# Cat-qubits: exponential protection against bit-flips




R. Lescanne, Z. Leghtas et al., Nature Physics, 2020

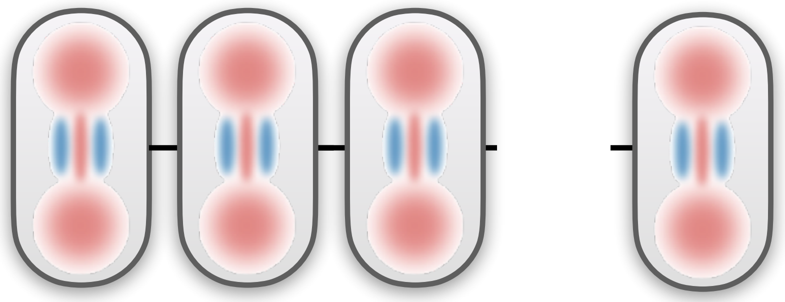


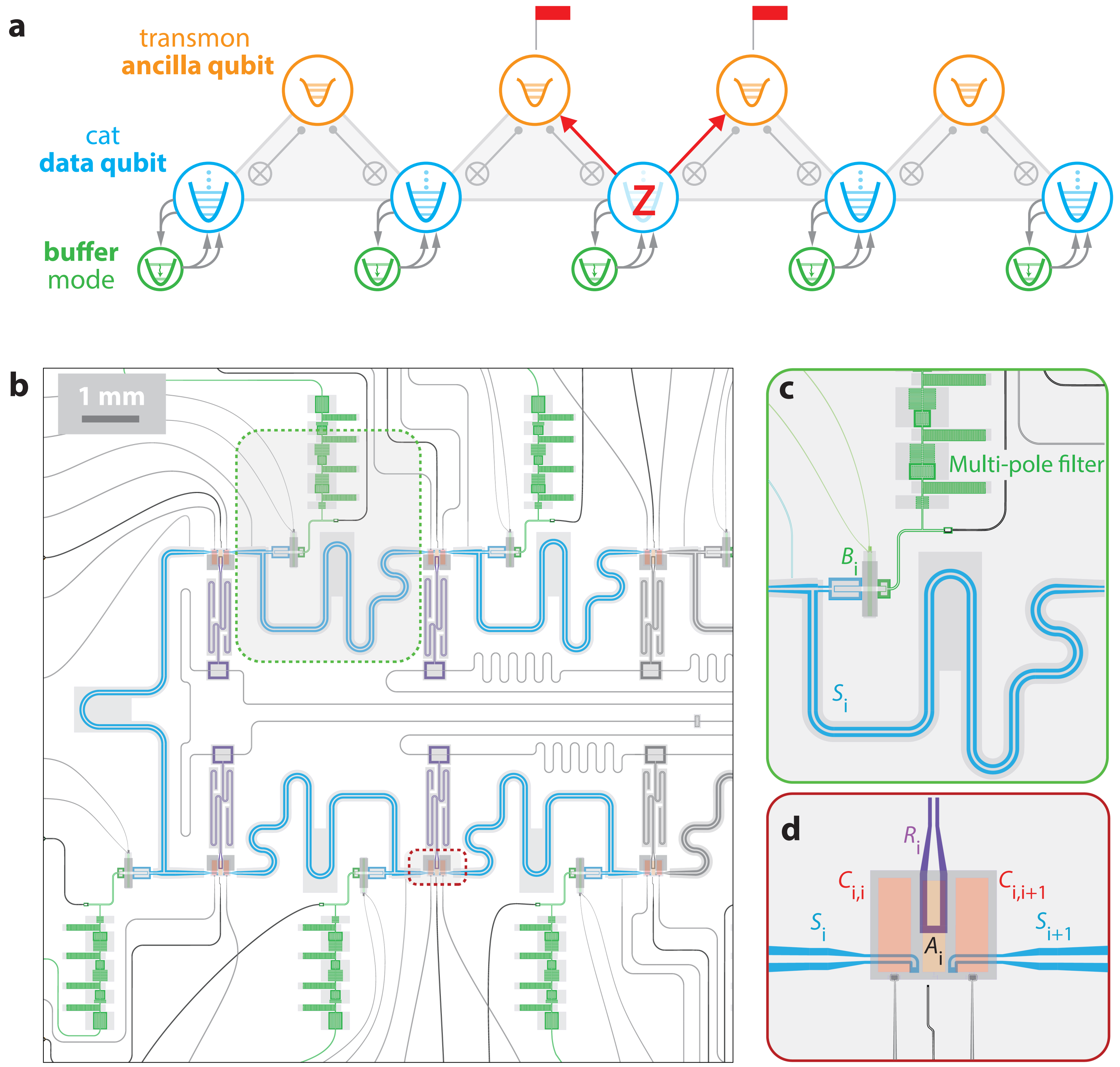

U. Reglade, Z. Leghtas et al., Nature, 2024.

# Fault-tolerance roadmap

- Large noise bias regime ($\bar{n} > 10 - 15$ photons): repetition code against phase-flips may be sufficient

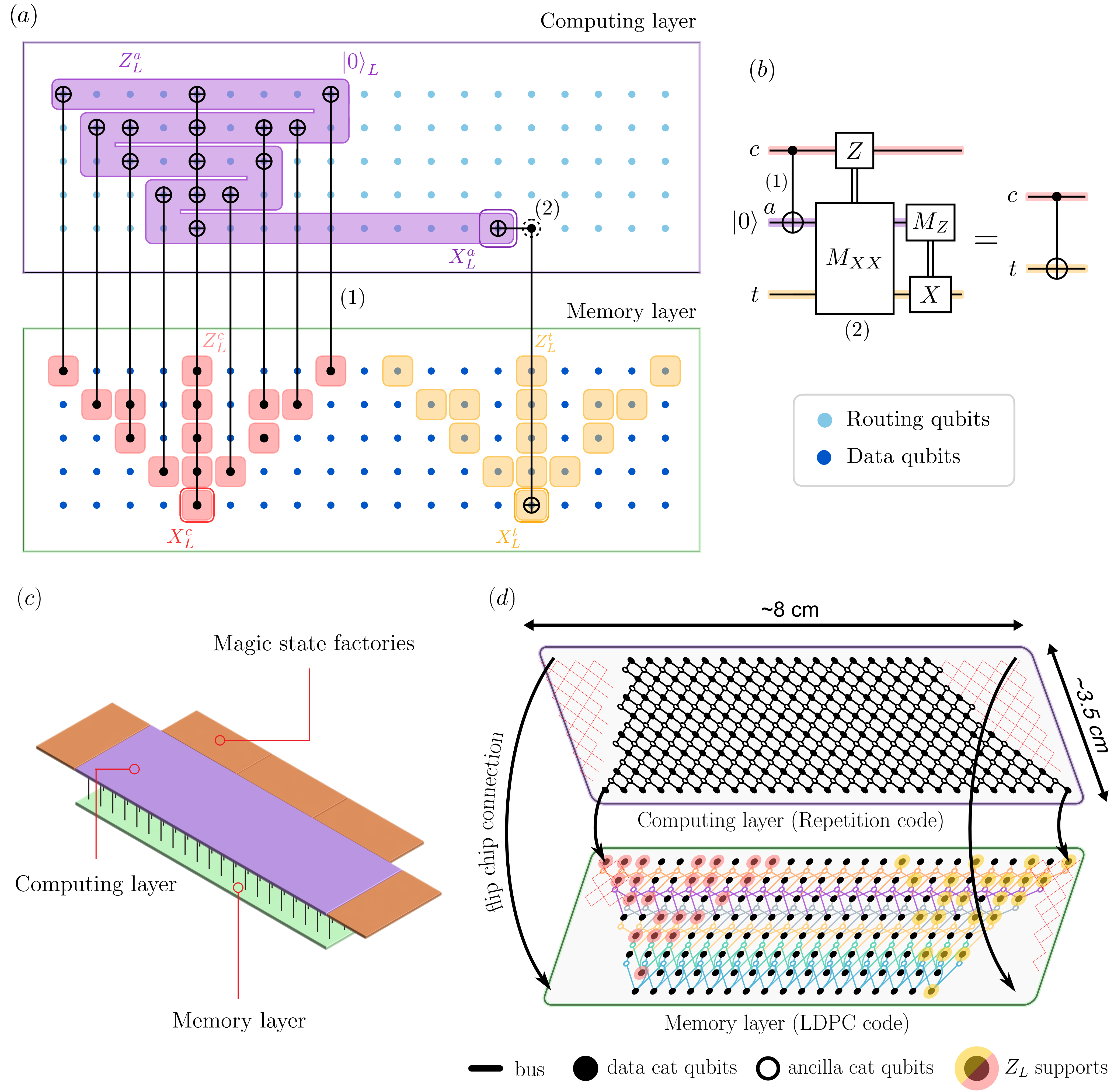


J. Guillaud and M. Mirrahimi, PRX 9, 041053, 2019




AWS experiment: Putterman, Painter et al., Nature, 2025.

2D-local architecture for LDPC-cat qubit concatenation and fault-tolerant operations




FT overhead: 758 cat-qubits with physical error rate $10^{-3}$ (per qubit, per gate) to encode 100 logical qubits at error rate $10^{-8}$ (compared to 33700 for surface code)

D. Ruiz et al., Nature Comm., 2025